Geodesic Based Conformal Mesh Parameterization

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ABSTRACT

In this paper, a novel method of conformal parameterization for triangular meshes is presented. Firstly, based on geodesic on a mesh, an algorithm constructing local barycentric coordinates is proposed. Then, these local coordinates are merged via a linear system to form a global conformal parameterization of the mesh. The conformal mesh parameterization method here can be viewed as a development of the shape-preserving method proposed by M. S. Floater. It avoids error of locally approximating the so-called geodesic polar mapping and hence giving better results. Experimental results are given to illustrate the effectiveness of proposed methods.

Keywords: Conformal parameterization, geodesic barycentric coordinates, triangular mesh, convex linear interpolation.

1. INTRODUCTION

Surface parameterization is to find a bijective mapping between two surfaces with similar topology. If these surfaces are discretized as meshes, the problem of computing such piecewise mapping is known as mesh parameterization. Therefore, for a disk-like triangular mesh, its parameterization is to find a planar counterpart that represents it. Parameterization of spatial meshes has numerous applications in computer graphics and CAD/CAM, probably because it allows some 3D geometric processing to be operated on simple domains, e.g., planar or sphere. For instance, texture mapping, remeshing and tool-path planning for machining, to name a few. In practice, those parameterizations having geometric properties of interest (e.g., conformal, isometric and authalic) are preferable. Isometric ones are maybe perfect for almost every application. However, these parameterizations only exist for few surfaces. Therefore, the conformal parameterization is the second best choice, since it preserves the shape locally with respect to a scale factor and can be obtained for most surfaces.

Among surfaces to be parameterized, those with disk-like topology are fundamental and more valuable for many applications. The triangles in parameter domain will typically be slightly different from that of spatial mesh, resulting in angle and length distortion. Thus, one way to parameterize a 3D mesh is to minimize such distortion in a least square sense. There are several methods presented in the literature to measure the angle distortion, resulting in different conformal parameterization. The ABF method as well as its developments \cite{1, 2, 3} flattens a 3D mesh onto a plane by minimizing a distortion metric describing the difference of corresponding angles between the 3D triangles and 2D ones. These methods require intensive computation because the distortion measuring is non-linear. It is known that a mapping is conformal if and only if it satisfies the Cauchy-Riemann equations. Lévy et al. \cite{4} obtained a conformal parameterization by minimizing an energy integral based on the Cauchy-Riemann equations. Desbrun et al. \cite{5} first derived the gradient of an angle on a 3D mesh with respect to its adjacent vertex, then constructed a quadratic energy based on such gradient, and finally taken critical point of the energy as a parameterization of the mesh. In fact, when moving a vertex of the mesh, it will change not only the adjacent angles but also angles nearby. Therefore, minimizing the energy won’t obtain an accurate conformal parameterization. The other way to compute a conformal parameterization resorts to harmonic mapping. The advantage of harmonic mappings over conformal ones is the ease with which they can be computed \cite{6}. Pinkall et al. \cite{7} first discretize harmonic mappings over a mesh based on the Finite Element Method (FEM) and then Eck et al. \cite{8} intuitively shown that the discrete mapping in \cite{7} minimizes some spring energy. Floater proposed the mean value coordinates method in \cite{9, 10}. The methods of Floater and Eck require the boundary of the mesh to be pre-fixed. In the literature, these methods are also classified as barycentric mapping theorem based algorithms. However, according to the Riemann mapping theorem, a conformal mapping and a harmonic mapping are not equivalent, i.e., if a mapping from a 3D surface to a planar region is conformal, it is harmonic too, but not vice versa \cite{6}. Additionally, some recent works on conformal parameterization can be referred in \cite{11, 12, 13}.
For disk-like triangular meshes, another conformal parameterization method is developed in this paper from the one presented by M.S. Floater [14]. His basic idea is that each vertex is expressed as a weighted linear combination of its neighbor vertices. Parameterizations of different properties can be obtained by selecting different combination weight schemes. He proposed a shape-preserving weight scheme based on the conformality of barycentric coordinates which is furthermore based on [15]. As opposed to the original method, we build a geodesic based barycentric coordinates directly on meshes rather than on an intermediate tangent-like plane. The original projecting procedure will inevitably introduce error into parameterization, while, the proposed direct method is free from this.

2. CONFORMAL PARAMETERIZATION

Consider a triangular mesh \( M = (V, E) \), where \( V \) and \( E \) are vertex and edge sets respectively, representing a surface \( S \) embedded in \( \mathbb{R}^3 \). A mapping between a planar triangular mesh and \( M \) can be described by the correspondence between their vertices. Thus, parameterization of \( M \) can be modeled in the following form:

\[
f : V_2 \rightarrow V_3,
\]

where \( V_2, V_3 \) are the vertex sets of planar and spatial triangular mesh, respectively. The so-called conformal parameterization means that the mapping \( f \) preserves angles between the two meshes.

There are two crucial steps of our parameterization algorithm. The first step is to build local barycentric mapping expressing each vertex as a linear combination of its neighbors conformally. The second step is to merge the local mappings together to achieve a conformal parameterization for the whole mesh.

2.1 Local conformal mapping

M.S. Floater employed an intermediate projecting mapping, i.e., geodesic polar mapping, to construct the local barycentric coordinates [14]. With the help of such mapping, each vertex and its neighbor vertices can be mapped to a tangent plane at the vertex in an isometric sense. Subsequently, as known, constructing barycentric coordinates for planar points is trivial. Since the local mapping is isometric with respect to centroid vertex, it is conformal and authalic. The so-called geodesic polar mapping is a relation between neighborhood of a point and its tangent plane, which preserves geodesic distance locally. Its definition is as follows:

**Definition 1.** The geodesic polar mapping at a point \( p \) on a smooth surface \( S \) associates to each tangent vector \( v \) in the tangent space \( T_p \), a point on a geodesic \( C_s(\cdot) \) through \( C_s(0) = p \) with initial direction \( C_s'(0) = v \) as follows:

\[
f : T_p \rightarrow S, \quad f(v) = C_s(\|v\|)
\]

For a given vector \( v \in T_p \) the radial lines \( rv \subset T_p \), \( r \in \mathbb{R} \) are mapped isometrically to a geodesic ray \( C_s \).

The mapping implies a local isometric mapping over a local patch to the tangent plane along each radial direction. M.S. Floater exploited the method proposed by Welch and Witzkin [16] to approximate the geodesic polar mapping for triangular meshes. Consider a vertex and its 1-ring neighbor vertices, as shown in Fig. 1. Let \( p_i \) donates image point of vertex \( v_i \) and correspondingly for \( p_j \) and \( v_j \). By keeping the distances between \( v \) and \( v_i \) unchanging with

\[
\|p_i - p\| = \|v_i - v\|, \quad (3)
\]

and scale the angles of consecutive vertices around \( v \) as

\[
\text{Angle}(v_i, v_j) = \sigma \cdot \text{Angle}(p_i, p_j), \quad (4)
\]

where the scale factor \( \sigma \) forces sum of angles around center point \( p \) to be \( 2\pi \).

With the geodesic polar mapping in hand, barycentric coordinates for point \( p \) and its neighbors \( \{p_i\} \) can be easily computed. Floater first assigned a triangle, containing center point \( p \), for each point \( p_i \), as shown in Fig. 1. Subsequently, he expressed \( p \) as a convex combination of the triangle’s three vertices

\[
p = w_i p_i + w_j p_j + w_i p_i
\]

where the coefficients \( w_i \) are

\[
w_i = \frac{\text{Area}(\Delta p_i, p_j)}{\text{Area}(\Delta p, p_i)};
\]
By applying the previous procedure to each neighbor point $p_i$, we can get more than one coefficient for $p_j$. Floater constructed local conformal mapping by taking average of these coefficients

$$p = \sum_{p_i \in N(p)} \lambda_i p_i,$$

where $N(p)$ is the 1-ring neighborhood.

![Figure 1. Geodesic polar mapping and barycentric coordinates.](image)

However, the uniformly scaling procedure of angles is unreasonable, which will inevitably introduce error when parameterizing. We next propose a direct barycentric coordinates generating method on triangular meshes. As mentioned, the geodesic polar mapping preserves distances along each radial direction. Therefore, we can exploit the geodesic distances on meshes to construct barycentric coordinates rather than projecting local vertices onto tangent planes. The first procedure to construct such coordinates is the computation of geodesic distances on triangular meshes. Generally, piecewise lines on triangular facets are approximations of geodesic on a surface. Take the local mesh as shown in Fig. 2 (a) for example. Flatten triangles $\Delta v_j v_i$ and $\Delta v_i v_j$ onto some plane along their intersection $v_j$, resulting in a quadrilateral as shown in Fig. 2 (b). Thus, the geodesic curve from $v_j$ to $v_i$ is the piecewise lines corresponding to planar segment $v_j' v_i'$, as shown in Fig. 2 (a). If there are more than two triangles between $v_j$ and $v_i$, computing of geodesic between them is similar. And if point $v'$ is inside triangle $v_j' v_i'$, which means that spatial vertex $v$ is outside surface triangle $\Delta v_j v_i v_j$, the barycentric coordinates of the three vertices with respect to $v$ are meaningless. Therefore, for boundary vertices, all the barycentric coordinates of its neighbor vertices are set to be ZERO, implying that there is no conformal structure along the boundary.

After obtaining geodesic on meshes directly and assigning each neighbor vertex (e.g., $v_j$) with a surface triangle (e.g., polygon $v_j v_i v_j v_i v_j v_j$) containing center vertex $v$, we can construct barycentric coordinates as follows

$$p = w_j p_j + w_i p_i + w_o p_o$$

where the coefficients $w'_i$ are

$$w'_j = Area(\Delta v_j v_i) / A;$$

$$w'_i = Area(\Delta v_i v_j + \Delta v_j v_i) / A;$$

$$w'_o = Area(\Delta v_o v_j + \Delta v_j v_o + \Delta v_o v_i + \Delta v_i v_o) / A;$$

$$A = Area(\Delta v_j v_i + \Delta v_i v_j + \Delta v_j v_o + \Delta v_o v_j + \Delta v_i v_o + \Delta v_o v_i)$$
By letting \( w' = 0 \), if \( r \neq i, j, l \), Expression (7) can be rewritten as
\[
p = \sum_i w'_i p_i
\]
Eventually, take average of coefficients associated with \( v_j \) as its local conformal mapping weight
\[
w_i = \frac{1}{\text{size}(N(p))} \sum_j w'_j
\]
That is
\[
p = \sum w_i p_i \quad \text{and} \quad w_i > 0, \sum w_i = 1
\]

![Figure 2. Direct constructing of barycentric coordinates](image)

The essence of Floater’s method is to map a vertex and its neighbors onto the tangent plane as isometrically as possible. And then, merge those tangent planes to be a single plane conformally with barycentric coordinates. Our approach employed geodesic on meshes to compute the coordinates directly, which can avoid the intermediate error of approximating geodesic polar mapping and simplifies computation of parameterization.

### 2.2 Global conformal mapping

Since local conformal mappings for triangular mesh \( M \) has been computed, they can be adhered together by solving a linear system, resulting in a global conformal parameterization for \( M \).

Supposing that mapping \( f : M \rightarrow \mathbb{R}^2 \) is the inverse mapping of conformal parameterization, the linear system can be constructed as follows
\[
u_i - \sum_{v_j \in N(v_i)} w_j u_j = 0, \quad i = 1, \ldots, n
\]
where \( u_i \) is the corresponding planar point for mesh vertex \( v_i \), i.e., \( f(v_i) = u_i \). By letting \( w_id = 0 \) if \( v_i \) is not adjacent to \( v_j \), the linear equations (11) can be written in a matrix form as
\[
WU = 0
\]
where unknown vector \( U_{m+2} = (u_1, \ldots, u_n)^T \). For vertices that are not on boundary, its corresponding row in coefficient matrix \( W \) has diagonal entry \( w_{id} \) be 1. While for boundary vertices, its corresponding row is a ZERO row, which makes unknowns corresponding to these boundary vertices free variables. Namely, during conformally parameterizing a triangular mesh, boundary mapping can be pre-set. Donate boundary vertices set of mesh \( M \) by \( \partial M = \{v_{m+1}, \ldots, v_n\} \) and boundary mapping by
\[
f(\partial M) = (f_{m+1}, \ldots, f_n) = (u_{m+1}, \ldots, u_n) \subseteq \mathbb{R}^2
\]
Then, equations (12) can be rewritten as
\[
WU = 0 \Rightarrow AU^l = -BU^o
\]
where
\[
W = \begin{bmatrix}
A_{m\times m} & B_{m\times (n-m)} \\
C_{(n-m)\times m} & 0
\end{bmatrix}
\]
\[
U^l_{m+2} = [u_1 \cdots u_m]^T \quad \text{and} \quad U^o_{(n-m)+2} = [u_{m+1} \cdots u_n]^T
\]
Since the proposed weights are none negative, this linear system has a unique solution with boundary mapping \( f(\partial M) \) given. That is, coefficient matrix \( A \) is non-degenerated and
\[
U^1 = -A^{-1}B\Omega
\tag{15}
\]
Generally, the matrix \( A \) is sparse. Thus, the GMRES algorithm is utilized to solve linear equations (14).

Mapping between planar points and spatial mesh vertices induces mapping between triangular facets
\[
\varphi : t_i \rightarrow \overline{t_i}, \quad t_i \in T, \overline{t_i} \in \overline{T}
\tag{16}
\]
where \( T \) and \( \overline{T} \) donate triangular facets of triangular mesh \( M \) and planar mesh \( D \), respectively. Let
\[
\varphi'_i = \begin{cases} 
\varphi^{-1} & u \in \overline{t}_i \\
0 & u \not\in \overline{t}_i
\end{cases}
\tag{17}
\]
Thus, mapping \( \varphi = \sum \varphi'_i : D \rightarrow M \) is a piecewise approximating of some conformal parameterization for smooth surface \( S \) linearly represented by \( M \).

3. EXPERIMENTAL RESULTS

In this section, we implement the proposed algorithms on some real data. Two typical models are chosen to show the effectiveness of them, as shown in Fig. 3. One is a human face generated by a coordinate measuring machine, as shown in Fig. 3 (a). The other is a freeform surface generated by the UG software, as shown in Fig. 3 (b).

Boundary mapping should be set before solving Equations (14) and in this paper boundary vertices are assigned along the planar boundary according to chord length between adjacent vertices on the 3D boundary. Fig. 3 (c) shows the conformality (i.e., angle-preserving) property of the proposed algorithm, with its parameter domain set to be a disk (i.e., 3D boundary is mapped to be a planar circle). Fig. 3 (d) shows the conformality property of the proposed algorithm, with its boundary mapped to be a planar rectangle. Orthogonal curves on both figures demonstrate that right angles are preserved well between planar and spatial meshes. Fig. 3 (e) is presented to show comparisons between proposed method and the original method. It can be obviously shown that our method can give better results, especially for complex regions (e.g., regions near nose, eyes and forehead).
4. CONCLUSION

In this paper, an algorithm for conformally parametering triangular meshes originated from the M.S. Floater’s famous shape-preserving parameterization is proposed. Local barycentric coordinates is firstly constructed based on geodesic on meshes, as opposed to the original method needing to project local patches onto tangent planes. Since approximating such projecting mapping will inevitably introduce error into parameterization, our direct method is theoretically better. Then a linear system is utilized to merge each local barycentric coordinates as a global and conformal parameterization of triangular meshes. As known, solving sparse linear equations is rather simple. The proposed method can not only improve the angle-preserving (also called shape-preserving) of parameterization theoretically but also empirically shown in the comparison of Fig. 3 (d) and (e).

REFERENCES