1 Introduction

The study on structural synthesis methods for planar mechanisms has continuously being making progress. The introduction of graph theory and adjacency matrix particularly the computer-aided analysis greatly improves the structural synthesis which initially started from observation. The structural synthesis focuses on the structural topology and its variety. Davies and Crossley [1] proposed a structural synthesis method based on Frank’s condensed notation. Hass and Crossley [2] provided a synthesis method by adding binary links. Manolescu and Mruthyunjaya [3–7] used Baranov trusses’ transformation of binary chains for structural synthesis. Rao and Deshmukh [8] developed a method of computer-aided structural synthesis of planar kinematic chains based on loop formation. Chu and Cao [9,10] used an improved graph to study type synthesis of 1DOF and 2 DOF linkages, including the use of component group or module of the AG [11], and the use of dyads was developed by Lulian and Dan [12]. Recently, Assur graphs were used by Shai [13] in topological synthesis. In addition to topological approaches, Galletti [14] provided a modular approach for planar linkage analysis, Tuttle and coworkers [15–18] derived new mechanisms by distributing binary links among polygonal link bases, Li [19] developed a structural synthesis approach for spatial mechanisms. Usually, an adjacency matrix is used to describe the structural topology of planar mechanisms and provides connection information between links. Such use of an adjacency matrix is convenient for the computer process but does not contain enough information in building kinematic chains intuitively. Hwang and Hwang [20] used contracted-link adjacency matrices for the structural synthesis, Rao and Raju [21] used a hamming number to detect isomorphism in the structure synthesis, Li et al. [22] used a loop-link-matrix in the mechanism synthesis and isomorphism detection. Further, Yan and Kuo [23] developed a topological matrix for variable kinematic pairs, Li and Dai [24] developed the AG combinatorial theory and applied this concept to metamorphic mechanisms, proposing the concept of AAG. The structural combinatorial theory based on the AG connects AG with the driving links, and/or other AG and the frame link, making the synthesis process intuitive, convenient and practical. However, for the existing approach of using AGs for synthesis, the method is still discretionary and not consistent. Therefore, the result does not contain the whole solution.

This was also the case for structural synthesis of the metamorphic mechanism by focusing on designing 1DOF mechanisms for the task requirements of corresponding working configurations, followed by combining all 1DOF mechanisms to form the multidegrees of freedom source metamorphic mechanisms. The method has not been unified with the synthesis approach for conventional planar mechanisms. This paper attempts to unify the approach for both mechanisms, explores the group-based adjacency matrix in describing the complex connection, and obtains the comprehensive range of connection forms of metamorphic mechanisms which meet the given conditions.

The current challenge for synthesizing a metamorphic mechanism is to generate each multiple topological form of the mechanism, and to finally take a union of these forms. The proposed approach in this paper is to directly synthesize the source form of the metamorphic mechanisms by utilizing the proposed AAGs. This simplifies the process and obtains the synthesis result straightforward.

The previous study [25] established the relationship between a topological description and a mechanism, and constructed the topological matrix from a given mechanism, Ref. [26] made a preliminary study of the structural synthesis, and examined its
potential used for a planar mechanism. On the basis of the study in Refs. [25,26], this paper significantly updates the adjacency matrix as a group-based adjacency matrix by clarifying connectivity between links and joints with their properties, proposes a systematic approach with a detailed procedure and extends the synthesis to generating metamorphic mechanisms with multiple configurations. This leads to case studies of generating more connection forms than previous literature and obtaining a total of 588 kinds of mechanisms in the synthesis.

2 Notation of Elements in AGs

2.1 Elements of AGs

2.1.1 Notation of Connection Nodes in AGs. Notation of elements in AGs is defined in Fig. 1 by using a single letter as i to represent external pairs of the i link of the AGs, and using two letters ij to represent internal pairs between the i link and j link, and two same letters ii to represent the external nodes in the i link. The external nodes are to be used for connecting the external pairs during the synthesis.

2.1.2 Notation of Linkages for Class II AG. Class II AG is the most widely used kinematic chain, and has five types. According to the notation defined in Sec. 2.1.1, notation for the links in the kinematic chain for this AG can be defined in order to guarantee the uniqueness of notations for the pairs. The rules to notate the linkages are that the number of the links increases continuously from external pairs (nodes) to internal pairs. As shown in Fig. 2, $A_{II}^t$ ($t = 1, 2, ...$) is used to present the type of the kinematic chains of class II AG.

2.1.3 Classes III and IV AGs. It is well known that the class III AG is formed by one ternary link connected to three binary links with R and/or P pairs. The class IV AG is formed by two ternary links connected to two binary links with R and/or P pairs. The class III AG and the class IV AG kinematic chains in which all joints are R pairs are shown in Figs. 3(a) and 3(b), respectively.

2.2 Augmented AGs

2.2.1 Notation for the Linkages of Class II AAG. If an additional binary link and an R/P pair are inserted into class II AG in Fig. 2, mobility of an AAG becomes one instead of the zero in the original class II AG, and this AAG is called class II AAG in the paper. The group structures of the class II AAG should have 12 types in accordance with permutation and combination of three binary links and four R and/or P pairs. However, two of them are isomorphic, and one is formed with four P joints with mobility two; therefore, the class II AAGs has practically nine types of group structures. For any type of class II AAG, it is noted as $A_{II}^t$ ($t = 1, 2, ...$). Notations of the links and types of class II AAG are indicated in Fig. 4.

2.2.2 Classes III and IV AAG. If an additional binary link and a R/P pair are inserted into class III AG and class IV AG in Fig. 5, mobility of the group will become one instead of the zero in the original class III AG or class IV AG, and this AAG is called class III AAG or class IV AAG. The class III AAG and class IV AAG in which all the joints are R pairs are shown in Figs. 5(a)–5(c), respectively.

2.3 Notation for Driving Link and Frame Link. Figure 6 gives notation for the driving link and the frame link for both rotation and translation forms. The driving links are notified as $D^1$ and $D^2$, respectively.

3 Group-Based Adjacency Matrixes for Planar Mechanisms

3.1 Definition of the Group-Based Adjacency Matrix. In order to express the structure of a mechanism conveniently and
At the frame link of a mechanism, type of groups. For example, \( A_{\text{AG}} \) or class III \( A_{\text{AG}} \), ...; the superscript \( \text{AG} \) or class III \( \text{AG} \), ...; and \( c = 1 \) or 2, ... stands for the driving type as defined in Fig. 6, \( F \) is the frame link of a mechanism, \( A_{\text{AG}}^i \) is the class and type of the AGs and/or AAGs, in which, \( c = \text{II} \) or III, ... stands for class II AG or class III AG, ...; and \( p = 0 \) or 1, ... stands for class II AAG, or class III AAG, ...; the superscript \( t = 1 \) or 2, ... gives the type of groups. For example, \( A_{\text{AG}}^1 \) is type 1 class II AG, according to the definition in Fig. 2 that is the RRR AG, and \( A_{\text{AG}}^3 \) is type 2 class II AAG. \( J_p^i \) stands for the connection relationship and type of the joint. \( J_p^i \) is the type of joints which are between the diagonal elements (such as \( R, \ P, \ ...) \), \( k \) (or \( k' \)) is the joint number of the external pair for the next diagonal element, and \( p \) is the connection location of \( k \) (or \( k' \)) with the previous element. Following the notation described earlier, when \( p = i, \ p \) represents an external pair of the previous element. When \( p = ij, i \neq j, \ p \) represents an internal pair. When \( p = ij, i = j, \ p \) represents the external node that is to be connected to an external pair during the synthesis. When \( p = 0, \) it indicates that it is connected to the frame link of the mechanism. Further, \( J_p^i \) is 0, if there is no connection between two elements.

\[
A = \begin{pmatrix}
F & \cdots & J_p^1 & \cdots & J_p^t \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
J_p^t & \cdots & A_{\text{AG}}^l & \cdots & J_p^t \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
J_p^t & \cdots & J_p^t & \cdots & A_{\text{AG}}^l
\end{pmatrix}
\]

(1)

In the above matrix, elements in the diagonal entries present three fundamental elements of the mechanism, as the frame link, the driving link, and the AGs and/or AAGs. The off-diagonal entries express the connection relationship and type of joints between the three fundamental elements in the diagonal entries. In the matrix, \( D^l \) represents the driving link where the superscript \( l = 1 \) or 2, ... stands for the driving type as defined in Fig. 6. \( F \) is the frame link of a mechanism, \( A_{\text{AG}}^i \) is the class and type of the AGs and/or AAGs, in which, \( c = \text{II} \) or III, ... stands for class II AG or class III AG, ...; and \( c = \text{II} \) or III, ... stands for class II AAG, or class III AAG, ...; the superscript \( t = 1 \) or 2, ... gives the type of groups. For example, \( A_{\text{AG}}^1 \) is type 1 class II AG, according to the definition in Fig. 2 that is the RRR AG, and \( A_{\text{AG}}^3 \) is type 2 class II AAG, \( J_p^i \) stands for the connection relationship and type of the joint. \( J_p^i \) is the type of joints which are between the diagonal elements (such as \( R, \ P, \ ... \)), \( k \) (or \( k' \)) is the joint number of the external pair for the next diagonal element, and \( p \) is the connection location of \( k \) (or \( k' \)) with the previous element. Following the notation described earlier, when \( p = i, \ p \) represents an external pair of the previous element. When \( p = ij, i \neq j, \ p \) represents an internal pair. When \( p = ij, i = j, \ p \) represents the external node that is to be connected to an external pair during the synthesis. When \( p = 0, \) it indicates that it is connected to the frame link of the mechanism. Further, \( J_p^i \) is 0, if there is no connection between two elements.

\[
A_{\text{D1}} = \begin{pmatrix}
F & R_{01}^1 & D^1 \\
& A_{\text{II}}^3 & A_{\text{II}}^3 \\
& A_{\text{II}}^3 & A_{\text{II}}^3
\end{pmatrix}
\]

\[
A_{\text{D2}} = \begin{pmatrix}
F & R_{01}^2 & D^1 \\
& A_{\text{II}}^3 & A_{\text{II}}^3 \\
& A_{\text{II}}^3 & A_{\text{II}}^3
\end{pmatrix}
\]

\[
A_{\text{E2}} = \begin{pmatrix}
F & D^1 \\
& A_{\text{II}}^3 & R_{22}^1 \\
& A_{\text{II}}^3 & A_{\text{II}}^3
\end{pmatrix}
\]

In order to have a more intuitive view, the above matrices only display the connected elements and remove all zero elements. It can be noted that the group structures of a mechanism is described from the frame link to the higher level AG with the corresponding matrices step by step.

3.2 Examples in Building the Group-Based Adjacency Matrix.

3.2.1 Adjacency Matrix for 1DOF Mechanisms. There is a 1DOF 6-bar mechanism in Fig. 7, According to the composition principle of mechanisms based on AGs, it is assembled with one driving link, one frame link, and two class II AGs for RPR and RRP linkages.

Based on the notation for the AGs and the definition of the group-based adjacency matrix, this mechanism is divided into \( D^1 \) and \( F \) and two class II AGs, \( A_{\text{II}}^3 \) and \( A_{\text{II}}^3 \). According to the notation rules stated previously, the links and joints, the driving link, the frame link and the AGs can be illustrated in Figs. 8(a)–8(c), respectively.

Thus, the group-based adjacency matrices for the linkage groups of the mechanism can be established following the driving link, the frame link and the two AGs as:

\[
A = \begin{pmatrix}
F & R_{01}^1 & R_{01}^2 & P_{01}^2 \\
& D^1 & R_{11}^1 & 0 \\
& 0 & 0 & R_{22}^1 \\
& 0 & 0 & 0
\end{pmatrix}
\]

(2)

In the above group-based adjacency matrix, \( R_{01}^1, R_{01}^2, R_{11}^1, \) and \( R_{22}^1 \) represent the revolute pairs \( A, B, C, D \) of the mechanism as in Fig. 7, and \( P_{01}^2 \) represents the prismatic joint \( E \).
3.2.2 Group-Based Adjacency Matrix for Metamorphic Mechanisms.

Figure 9 gives a 2DOF metamorphic mechanism. Applying the composition principle of the metamorphic mechanism based on the AAGs, the mechanism in Fig. 9 is composed of a driving link, the frame link and one 1DOF RRRR AAG.

Base on the rules given previously, this mechanism can be split into the driving link \( D_1 \), the frame link \( F \) and a class II AAG as \( A_{\text{AII}} \). The number and notation for the driving link, the frame link and AAG as \( A_1 \) are given in Figs. 10(a) and 10(b).

The group-based adjacency matrices following three fundamental elements of the driving link, the frame link and the AAG for the linkage groups of the mechanism in Fig. 10 can be obtained as

\[
A_D = \begin{pmatrix}
F & R_0^1 \\
D^1 & A_{\text{AII}}^1
\end{pmatrix},
\]

\[
A_{A_{\text{AII}}} = \begin{pmatrix}
F & R_0^1 \\
D^1 & A_{\text{AII}}
\end{pmatrix} = \begin{pmatrix}
F & R_0^1 \\
D^1 & R_{11}^1
\end{pmatrix}.
\]

Integrating matrices \( A_D \) and \( A_{A_{\text{AII}}} \) and setting matrix elements in which nonconnections are set to 0, the group-based adjacency matrix for this metamorphic mechanism is

\[
A = \begin{pmatrix}
F & R_0^1 & R_0^2 \\
0 & D^1 & R_{11}^1 \\
0 & 0 & A_{\text{AII}}
\end{pmatrix}.
\]

4 AG Inferred Structural Synthesis

Following the composition principle of planar mechanisms using AGs and/or AAGs and applying the group-based adjacency matrix, the AG inferred structural synthesis is stated as follows. Choosing a link as the driving link \( D \) and connecting it to the frame link \( F \), followed by adding class \( c \) type \( t \) AG \( A_c^t \) and/or AAG sequentially, the AG/AAG can be connected to the external note of the driving link or the frame link, or connected to the previous AG/AAGs.

4.1 Relationship Between Number of the Links and AGs

(1) With the driving link and frame link each having one link, class II AG having two links, and class III and class IV AG having four links, etc., the relationship between the link number \( n \) and number of AGs of one-degree of freedom mechanisms can be established as follows:

\[
n = (2n_{\text{II}} + 4n_{\text{III}} + 4n_{\text{IV}}) + 2
\]

where \( n_i \) stands for the link number of an AG, when \( i = \text{II, III, IV}. \)

(2) With number of the AAGs of metamorphic mechanisms relating to the degree of freedom except for the driving link and let \( m_{\Lambda} \) stands for the number of the AAG, it follows that

\[
m_{\Lambda} = M - 1
\]

where \( M \) is the degree of the freedom of the metamorphic mechanism. With class II AAG having three links, class III and class IV AAG having five links, the relationship between link number \( n \) of the metamorphic mechanism and the number of the AAGs can be established as

\[
n = (3n_{A_{\text{II}}} + 5n_{A_{\text{III}}} + 5n_{A_{\text{IV}}}) + 2
\]

where \( n_{\Lambda_i} \) stands for the link number of an AAG, \( i = \text{II, III, IV}. \)

4.2 Group Permutation and Combination of \( A_c^t \). According to the permutation and combination theory, the permutation of taking \( r \) number from \( m \) number \( A_c^t \) can be presented as

\[
P_{n_c}^r = \frac{m!}{(m-r)!}.
\]
where the possible values of $r$ is $r = m - 1, m - 2, ..., 1$. Hence, the entire list of group permutation and combination of $A_i'$ for the diagonal elements of the group-based adjacency matrix is

$$ T = \sum_P P_m^r $$

(7)

### 4.3 Procedure of Group-Inferred Structural Synthesis

Constructing the group-based adjacency matrix progressively and synchronizing it with the isomorphism detection, the procedure of the group-inferred approach can be stated as follows:

1. Obtaining the assembly combination of the given linkage group: calculating link number and number of a list of group permutation and combination using Eqs. (3)- (7), obtaining permutation of $A_i'$ for the diagonal elements of the group-based adjacency matrix in Eq. (1), and synchronizing it with the isomorphism detection. Following this, the list of group permutation and combination $A_{ij}$ formed by the linkage groups can be obtained.

2. Obtaining driving-frame connection matrix $A_{Dj}$ with the same dimension as the final group-based adjacency matrix by only filling the matrix with the type of connection between driving link $D$ and frame link $F$.

3. Updating $A_{ij}$ to become subadjacency matrix $A_{ij}$, which expresses the connection relationship between first AG ($A_i')$ and the driving-frame connection matrix, and synchronizing it with the isomorphism detection.

4. Updating matrix $A_{ij}$ which expresses the connection relationship between AG ($A_i$), $i = 1, 2, ..., m$, and adjacency matrix $A_{ij} i < j$, synchronizing it with the isomorphism detection. The obtained $A_{ij}$ is a complete list of the connection forms for AG of $t = 1$.

5. Deriving structural forms which can be established by different types of linkages. Substituting $t = 2, 3, ...$ into $A_i^r$ which obtained according to connection types when $t = 1$, finding the corresponding mechanisms.

### 5 Structural Synthesis for 1DOF Mechanisms

This section is to find the entire list of structural forms of the planar mechanisms which have one-degree of freedom and the number $n$ of the links is six.

#### 5.1 Group Permutation and Combination for Given AGs

The one-degree of freedom, 6-bar planar mechanisms can be obtained using a driving link, the frame link and two class II AGs ($m = 2, r = 1$) or the linkage group composed of one class III AG and one class IV AG ($m = 1, r = 1$). With Eq. (7), the list of group permutation and combination of $A_i'$ can be calculated as

$$ T = \sum P_m^r = P_1^4 + 2P_1^3 = 4 $$

The group-based adjacency matrix which expresses this group combination can then be obtained as

$$ A_{F1} = \begin{pmatrix} 1 & D^t \\ A_{Ii1}^T & 1 \end{pmatrix}, \quad A_{F2} = \begin{pmatrix} 1 & D^t \\ A_{Ii2}^T & 1 \end{pmatrix} $$

In the meantime, isomorphism detection can be implemented as follows. Because the group combination only considers the assembly combinations of the AG components, it provides possible mechanisms connected only by revolute joints as hinges with $t = 1$, as

$$ A_{F2} = \begin{pmatrix} 1 & D^t \\ A_{Ii2}^T & 1 \end{pmatrix} = A_{F1} $$

Transforming and comparing the corresponding elements of $A_{F1}$ and $A_{F2}$, $A_{F2} = A_{F1}$ indicates that they are isomorphism mechanisms as

$$ A_{F2} = \begin{pmatrix} 1 & D^t \\ A_{Ii1}^T & 1 \end{pmatrix} $$

(1)

There are three forms of the group permutation and combination of the planar 6-bar hinge joint mechanisms. It should be notified that the above matrices only give the diagonal elements which express the information about the sequence of the linkage connection. This simplification is also applied in the following.

#### 5.2 Driving-Frame Connection Types

If the list of connection types of the driving link and the frame link is found, $A_{ij}$ can be established. Because this only gives out the connection relationship between $D$ and $F$, with type of joints supposed as $R$ pair, $J = R = 1$ in accordance with the definition of the group-based adjacency matrix.

Since $J = R = 1$, form $A_{ij}, i = 1, 2, 3$, the AG is symmetric, the joint number of the external joint for the next following element $k$ is 1 or 2 or 3 and connected to the frame link of the mechanism, that $p = 0$, the connection type between link $D$ and frame link $F$ of the planar 6-bar mechanism can be found as

$$ A_{D3} = \begin{pmatrix} 1 & D^t \\ A_{II}^T & 1 \end{pmatrix} $$

#### 5.3 Group Combination Between Arbitrary AGs With Corresponding Previous Components

Due to the composition principle of mechanisms with AGs, the following constraint conditions between linkage groups need to be met:

1. Numbers of the connection joints between the subsequent AG with the previous AG and/or the frame link and/or are the same with the number of external pairs of the subsequent AG.

2. External pairs of the linkage cannot be connected with the nodes in the same linkage.

#### 5.3.1 Connection Type Between Groups and Driving Link Extending to Frame Link

In compliance with the constraint conditions stated above, the connection type between all levels of AGs and elements in $A_{Dj}$ can be obtained following step 3 stated in Sec. 4.3 for establishing the group-based adjacency matrix.

For the planar 6-bar mechanisms, the connection type between the first linkage group and external pairs of the frame link and driving linkage groups (expressed as $A_{Dj}$) can be indicated as

$$ A_{II} = \begin{pmatrix} 1 & D^t \\ A_{II}^T & 1 \end{pmatrix} $$

where the joint number of the external joint for the subsequent group $k$ is 1 and 2, the connection location of $k$, i.e., $p$ is 0 and 11. The connection relationship between the first linkage group and $A_{Dj}, i = 2, 3$, can be found as
where $k = 1, 2, 3$ and $p = 0, 11$ in accordance with definition of the group-based adjacency matrix, and

$$A_{13} = \begin{pmatrix} F & 1 & 1 \\ D^i & 1 & 1 \\ A_{III}^1 & 1_{11} \\ A_{III}^1 & 1_{22} \\ A_{III}^1 & 1_{II2} \end{pmatrix}$$

where $k = 1, 2$ and $p = 0, 11$ for the adjacency matrix.

Examining configurations of the obtained mechanisms which have the connection type stated above, it can be found that there is no isomorphism mechanism. Hence, there are three independent connection relationships.

5.3.2 Group Permutation and Combination of AGs and Corresponding Previous Elements. Finding the connection relationship between $(A^1_i)_j$ and elements presented in the diagonal entries of the adjacency matrix in the previous level, the connection relationship of the corresponding elements are $(A^2_j)_i$ and $A_{II-1,j}$.

For the planar 6-bar mechanisms, the connection between the second AG $A_{II2}$ and $A_{II1}$ can be expressed as

$$A_{21} = \begin{pmatrix} F & 1 & 1 \\ D^i & 1 & 1 \\ A_{III}^1 & 1_{11} \\ A_{III}^1 & 1_{22} \end{pmatrix}$$

where $k = 1, 2$ and $p = 11, 22$.

$$A_{22} = \begin{pmatrix} F & 1 & 1 & 1 \\ D^i & 1 & 1 & 1 \\ A_{III}^1 & 1_{11} \\ A_{III}^1 & 1_{II2} \end{pmatrix}$$

where $k = 1, 2$ and $p = 0, 11$ or $p = 0, 22$.

$$A_{23} = \begin{pmatrix} F & 1 & 1 & 1 \\ D^i & 1 & 1 & 1 \\ A_{III}^1 & 1_{11} \\ A_{III}^1 & 1_{II2} \end{pmatrix}$$

Further, the group-based adjacency matrix representing all hinged 6-bar mechanisms with no composite hinges integrated with class III and class IV AGs of $m = 1$ can be obtained as

$$A_{II} = \begin{pmatrix} F & 1 & 1 & 0 \\ D^i & 1 & 1 & 0 \\ A_{III}^1 & 1_{11} \\ A_{III}^1 & 1_{II2} \end{pmatrix}$$

where $k = 1, 2, k' = 3$, and $p = 0, 11$.

Summarizing the group-based adjacency matrices stated above of $A_i, i = 1, 2, ..., 8$, there are eight connection forms obtained for this hinged 1DOF 6-bar mechanisms including inversions, with connection forms illustrated in Fig. 11. In using a traditional approach, Tuttle [18] obtained only five types of connection forms including inversions of kinematic chains. The difference is the group-based structural synthesis obtains three more forms based on the drive link selection.

Therefore, the group-based structural synthesis includes choice of the driving link with mechanism inversions. Using the same approach and following the same procedure, the structural synthesis of mechanisms which have more AGs can also be completed.
6 Structural Synthesis of Metamorphic Mechanisms

A metamorphic mechanism has more degrees of freedom with an ability of changing the topological configuration during motion, synthesis of metamorphic mechanisms is to use AAGs that contain more links and connection forms to directly synthesize the source form of metamorphic mechanisms by utilizing the proposed AAGs. This greatly simplifies the process and obtains the synthesis result straightaway.

6.1 Group Permutation and Combination of Given AG.
Using Eq. (4), number of the AAG can be calculated as

\[ m_A = M - 1 = 1. \]

In accordance with Eq. (5), 2DOF 5-bar mechanisms can be composed of the driving link, the frame link and a class II AAG \((m = 1, r = 1)\), and the 2DOF 7-bar mechanisms can be composed of the class III or the class IV AG \((m = 1, r = 1)\).

Calculating the number of the group permutation and combinations of the class I to class IV \(A_i^r\) linkage groups as

\[ T = \sum_p \begin{pmatrix} 5 \\ p \end{pmatrix} = 3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3 \]

The group-based adjacency matrices for the connecting relationships can be expressed as

\[ A_{T1} = \begin{pmatrix} F & D^1 \\ 1 & D^1 \end{pmatrix}, \quad A_{T2} = \begin{pmatrix} F & D^1 \\ 1 & D^1 \end{pmatrix}, \quad A_{T3} = \begin{pmatrix} F & D^1 \\ 1 & D^1 \end{pmatrix} \]

For isomorphism detection, it can be seen that there are no isomorphism mechanisms.

6.2 Driving-Frame Connection Type Between Driving Link \(D\) and Frame Link \(F\).

The entire connection relationship between driving linkage \(D\) and frame link \(F\) can be obtained based on the group-based adjacency matrix \(A_{T0}\), in which the driving-frame connection type between \(D\) and \(F\) can be obtained as \((k = 1, p = 0)\).

\[ A_{D1} = \begin{pmatrix} F & 1 \\ 1 & D^1 \end{pmatrix}, \quad A_{D2} = \begin{pmatrix} F & 1 \\ 1 & D^1 \end{pmatrix}, \quad A_{D3} = \begin{pmatrix} F & 1 \\ 1 & D^1 \end{pmatrix}, \quad A_{D4} = \begin{pmatrix} F & 1 \\ 1 & D^1 \end{pmatrix} \]

where \(A_{D3}\) and \(A_{D4}\) express two connection types of all \(R\) joints class IV AAGs in Fig. 5.

6.3 Connection Types Between AGs and Driving Link/Frame Link.
In accordance with step 3 in Sec. 4.3 and the constraint conditions presented above, the connection types can be obtained between all levels of linkage groups and links of \(A_{Dj}\).

Based on \(A_{D1}\), the connection type between \(A_{I1}\) and elements \(D\) and \(F\) is

\[ A_{11} = \begin{pmatrix} F & 1 & 1 \\ D^1 & 1 & 1 \end{pmatrix} \]
Based on $A_{D2}$, the connection type between $A_{VIII}^i$ and elements $D$ and $F$ is

$$A_{12} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i & A_{VIII}^i \\ 1 & 1 & A_{VIII}^i \end{pmatrix} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i \\ 1 & 1 & A_{VIII}^i \end{pmatrix}$$

For isomorphism detection, it was found there are two isomorphism mechanisms as

$$A_{13} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i & A_{VIII}^i \\ 1 & 1 & A_{VIII}^i \end{pmatrix} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i \\ 1 & 1 & A_{VIII}^i \end{pmatrix}$$

Further, based on $A_{D3}$, the connection type between $A_{VIII}^i$ and elements $D$ and $F$ is

$$A_{14} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i \\ A_{VIII}^i \end{pmatrix}$$

$$A_{15} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i \\ A_{VIII}^i \end{pmatrix}$$

Based on $A_{D3}$, the connection type between $A_{VIII}^i$ and elements $D$ and $F$ is

$$A_{16} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i \\ A_{VIII}^i \end{pmatrix} = \begin{pmatrix} F & 1 & 1 \\ D^1 & t_{11}^i \\ A_{VIII}^i \end{pmatrix}$$

This gives two isomorphism mechanisms.

Since $m = 1$ in this example, above 6 group-based adjacency matrices are generating the entire list of hinged 2DOF 5-bar or 7-bar mechanisms composed of AAGs

$$A_1 = A_{11} = \begin{pmatrix} F & 1 & 1 \\ 0 & D^1 & 1 \\ 0 & 0 & A_{VIII}^i \end{pmatrix}$$

$$A_2 = A_{12} = \begin{pmatrix} F & 1 & 1 \\ 0 & D^1 & t_{11}^i \\ 1 & 0 & A_{VIII}^i \end{pmatrix}$$

$$A_3 = A_{13} = \begin{pmatrix} F & 1 & 1 \\ 0 & D^1 & t_{11}^i \\ 0 & 0 & A_{VIII}^i \end{pmatrix}$$

$$A_4 = A_{14} = \begin{pmatrix} F & 1 & 1 \\ 0 & D^1 & t_{11}^i \\ 0 & 0 & A_{VIII}^i \end{pmatrix}$$

$$A_5 = A_{15} = \begin{pmatrix} F & 1 & 1 \\ 0 & D^1 & t_{11}^i \\ 0 & 0 & A_{VIII}^i \end{pmatrix}$$

$$A_6 = A_{16} = \begin{pmatrix} F & 1 & 1 \\ 0 & D^1 & 1 \\ 0 & 0 & A_{VIII}^i \end{pmatrix}$$

Thus there are only one connection form for the 5-bar metamorphic mechanisms and five types of connection forms for the 7-bar metamorphic mechanisms. The six types of connection forms of the 2DOF metamorphic mechanisms obtained from $A_i$, $i = 1, 2, ..., 6$, are shown in Fig. 12.

Therefore, the entire list of connection forms of the planar metamorphic mechanisms with two-degrees of freedom ($M = 2$) are obtained with 5-bar linkages or 7-bar linkages ($n = 5$ or $n = 7$).

### 7 Conclusions

This paper developed a group-inferred structural synthesis approach for both planar mechanisms and metamorphic mechanisms. Introducing the notation of AAGs, the paper established group-based adjacency matrix that firstly integrated the AGs/AAG in the adjacency matrix, paving a new way of setting a standard and convenient process for using AGs for structural synthesis, making the synthesis process more logical and uniform.

The paper then proposed group-inferred structural synthesis method for generating a complete list of mechanisms by considering an AG/AAG as a topological element and resorting to group-based adjacency matrices. A standard procedure was developed for this structural synthesis and relationships between parameters and number of synthesized mechanisms was established. This gave a way of group-inferred structural synthesis and presented a systematic way of structural synthesis for 1DOF planar mechanisms, leading to AAG-inferred structural synthesis of metamorphic mechanisms.

Based on the proposed approach, 588 mechanisms of the planar 1DOF 6-bar mechanisms were obtained with class II AGs, in which the driver link selecting and mechanism inversion are considered with both revolute pairs and prismatic joints. Applying the same approach, six connection forms of 2DOF 5-bar and 7-bar metamorphic mechanisms are obtained with class II, class III, and class IV AGs. The synthesis result is in the form of the group-based adjacency matrix and contains link information and connection types with the AGs/AAG. The new approach presents
advantages of maintaining structural characteristics of the AGs during the synthesis and paves a way for synthesis with AGs and AAGs for planar mechanisms and metamorphic mechanisms.

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