Abstract: This study proposes a novel sampling and control strategy to find a suboptimal sampling period sequence and control input sequence, such that a quadratic cost function of state and control input of a networked control system (NCS) with packet disordering is minimised. First, a discrete-time system model of the NCS with packet disordering, transmission delay and packet loss in terms of displacement values of packets is put forward. Second, a linear quadratic regulation (LQR) problem of the NCS is formulated, showing that the optimal controller depends on sampling period and quality of services (QoS) of networks. Interactive effects between sampling period and QoS of networks pose a challenge in solving the LQR problem of the NCS. To overcome this difficulty, different from traditional transmission-delay-based or packet-loss-based sampling scheme, a novel packet-disordering-based sampling period selection scheme is proposed. Furthermore, an algorithm is presented to find a suboptimal solution to the LQR problem in this study. Finally, simulation results demonstrate the effectiveness of the proposed approach.

1 Introduction

Facing the increasing demand of control networks in industrial processes, commercial systems and traffic systems, a rapidly increasing focus on network control systems (NCSs) has been witnessed [1–3]. NCSs have been hotspots of research in the past decades. The introduction of control networks can improve the efficiency, flexibility and reliability of these applications, reducing installation, maintenance, reconfiguration and costs. However, some unexpected phenomena potentially occur in network channels including transmission delay, packet loss and packet disordering and so on [4, 5]. Therefore, different from traditional control systems, optimal control problems of NCSs require not only control system design, but also network system design. In this paper, we investigate the effects of sampling rate and packet disordering on solution of linear quadratic regulation (LQR) problem.

We first formulate the LQR problem for the following continuous linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

(1)

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, respectively, are state and control input of the plant, and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are matrices with appropriate dimensions. The term $t_0$ is the initial time instant and $x_0$ is the initial state. Consider the following quadratic performance index for system (1)

$$J_T = \frac{1}{2} \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t)) \, dt$$

(2)

where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are positive semi-definite matrix and positive definite matrix, respectively, and the term $t_f$ is the terminal time instant.

The control input is assumed to be piecewise constant

$$u(t) = u_k \quad \forall t \in [t_k, t_{k+1})$$

(3)

where $u_k = u(t_k)$, $t_k$ is the $k$th sampling time instant with $0 < t_0 < t_1 < \cdots < t_k < \cdots \leq t_f$, and $h_k = t_{k+1} - t_k$ is called sampling period. We assume $h_{\operatorname{min}} \leq h_k \leq h_{\operatorname{max}}$, where $h_{\operatorname{max}}$ and $h_{\operatorname{min}}$ are the maximum and minimum values of the sampling period, respectively.

For given sampling periods, the optimal control input $u_k$ that minimises the cost function (2) for system (1) can be analytically determined through solving the discrete-time LQR problem (see [6–8])

$$\min_{u_k} \frac{1}{2} \sum_{k=0}^{N-1} (x_k^TQ_kx_k + u_k^TR_ku_k + 2x_k^TR_kW_y u_k + x_{k+1}^TQ_{k+1}x_{k+1})$$

(4)

s.t. \begin{align*}
  & x_{k+1} = A_kx_k + B_ku_k \\
  & x(t_0) = x_0
\end{align*}

(5)

where $x_k = x(t_k)$, $x_{k+1} = x(t_{k+1})$ and

$$A_k = \Phi(h_k), \quad \Phi(t) = e^{At}$$

(6)

$$B_k = \Gamma(h_k), \quad \Gamma(t) = \int_0^t e^{A\tau} \, d\tau B$$

(7)

$$Q_k = Q(h_k), \quad Q(t) = \int_0^t \Phi(t) Q(t) \Phi(t) \, dt$$

(8)

$$R_k = R(h_k), \quad R(t) = \tau R + \int_0^t \Gamma(t) Q \Gamma(t) \, dt$$

(9)

$$W_k = W(h_k), \quad W(t) = \int_0^t \Phi(t) Q \Phi(t) \, dt$$

(10)
Based on the optimal control theory \([6-9]\), from (4)-(10), one can easily find that the optimal control input sequence \(u_k (k = 0, 1, \ldots, N)\) depends on \(A_k, B_k, Q_k, R_k\) and \(W_k\) which are the functions of the sampling period \(h_k\). What’s more, for a closed-loop NCS with a digital controller shown in Fig. 1, the control input \(u_k\) is usually unavailable for the actuator at time instant \(t_k\) due to inherent transmission delay and packet loss caused by network communication. This means that \(u_{k-i} (i = 0, 1, \ldots)\) might act on the plant during the time interval \([t_{k-i+1}, t_k]\) (see \([4, 5]\)), i.e. the \(u_k\) in (5) should be changed into the \(u_{k-i}\). Thus, the optimal control input sequence \(u_k (k = 0, 1, \ldots, N)\) might be dependent on the sampling period and quality of services (QoS) of the network for the LQR problem of NCSs. In addition, it is well known that there exists a coupled relationship between the sampling period and QoS of the network. As the sampling rate gets higher, the network traffic load would become heavier, resulting in longer transmission delay and increased packet loss. The poor QoS of the network probably have negative impact on control performance of NCSs. While too slow sampling rate would destabilise NCSs since fewer packets are available to be used for updating the systems (see Fig. 2 \([10]\)). Thus, sampling rate should be considered when investigating the LQR problem of the NCS. However, how to select the sampling period and find appropriate control input sequence such that the performance index (2) is minimised is still an open issue so far \([10, 11]\).

1.1 Related works

It is well known that sampling period plays a pivotal role in design and synthesis of an NCS. Time-triggered communications so far are implemented by many NCSs \([12, 13]\). Generally speaking, this sampling approach might result in inefficient utilisation of limited network resources due to unnecessarily fast sampling rate \([14, 15]\). To reduce the sampling so as to save communication bandwidth, event-triggered samplings have been proposed \([14-16]\). In \([17]\), a robust self-triggered sampler with respect to external disturbance and transmission delay for ensuring globally ultimately uniform boundedness of every closed-loop system was presented. As shown in \([17]\), sampling period and network parameters should be jointly designed such that a good control performance of closed-loop NCSs can be achieved.

In \([10]\), the sampling periods corresponding to the points B and C in Fig. 2 were determined, while the analytical or numerical solution of the optimal sampling period of an NCS was unavailable. Dong and Kim \([11]\) presented near-optimal sampling periods by constructing the approximated exponential and quadratic functions of performance index against sampling period. Very recently, Bini and Buttazzo \([6]\) derived a necessary condition to achieve optimality of a set of sampling rates for the LQR problem of a sampled-data control system. However, the results obtained by Bini and Buttazzo \([6]\) are not effective to NCSs, since communication networks are not considered in \([6]\). To the best of the authors’ knowledge, optimal sampling and control strategies of an NCS have not been fully investigated in the existing literature so far, which motivates this study.

1.2 Main contributions

In this paper, we propose a sampling and control strategy for an NCS, such that a suboptimal solution is derived to minimise the quadratic cost function of state and control input. Packet disordering means that packets sent earlier may arrive at the destination node later or vice versa. Out-of-order arrival of packets at the destination is an increasingly common phenomenon in the internet \([18, 19]\). From the point of view of users, it is highly desired to increase the sampling rate and increase parallelism within routers and switches for improving control performance, unfortunately, all these point to future networks with increased packet disordering \([20]\). Packet disordering can lead to inefficient use of resources both within the network and at the end stations \([5, 18, 21-25]\), thus it has a significant impact on end-to-end performance of applications. Furthermore, sampling rate affects degree of packet disordering \([22]\). This is why different from the traditional transmission-delay-based and packet-loss-based sampling period scheduling schemes which have been more extensively studied (e.g. \([12-17]\)), we view packet disordering as a key index of QoS of a network and propose a novel packet-disordering-based sampling and control strategy for the LQR problem of an NCS.

The major contributions of this paper include two aspects. One is that the relationship between the sampling frequency and packet disordering in an NCS is clarified for the first time. The other is to propose a numerical sampling and control algorithm for providing a suboptimal solution to the LQR problem of the NCS with packet disordering compared with \([10]\).

This paper is organised as follows. The problem statement is presented in Section 2. In Section 3, the LQR problem of an NCS is formulated. Section 4 presents a suboptimal solution to the LQR problem of the NCS. Simulation results are presented in Section 5. Conclusions are drawn in Section 6.

2 Problem statement

In this section, we put forward a discrete-time system model for an NCS with packet disordering, transmission delay and packet loss. As shown in Fig. 1, packet disordering, transmission delay and packet loss are assumed to exist in the network channels. Bounded network transmission delays \((0 \leq \tau_{ac}^k + \tau_{ca}^k \leq m_1 h_{\text{mm}})\) and packet losses (the maximum consecutive packet loss is \(m_2\)) are naturally assumed in both channels, namely from the sensor to the controller and from the controller to the actuator, where \(m_1\) and \(m_2\) are some positive integers, and \(\tau_{ac}^k\) and \(\tau_{ca}^k\) denote the transmission delays of the \(k\)th packet transferred from the sensor to the controller and from the controller to the actuator, respectively. Set \(s = m_1 + m_2\).

We assume that the sensor and actuator are synchronously driven at each time instant \(t_k\) and the controller is event driven. Whenever information including sampled packet \(x_k\), parameters of networks and previous control input stored in the buffer of the actuator arrives at the controller, the controller begins to calculate the control signal, and then it is immediately sent to the actuator via the

Fig. 1 Architecture of an NCS
network. Without loss of generalisation, we let the subscript of control input $u_k$ correspond to that of received packet $x_k$ at the controller end. The time stamp is written in every data packet. Among the received control inputs by the actuator, the newest one is used to update the plant. Here, the newest control input means the one whose subscript is the nearest to the current sampling time instant.

It is certain that packet $u_{k-m_1}$ arrives at the actuator before and inclusive of the time instant $t_k$ if it is not lost due to $0 \leq t^i = \tau_{k_c}^i + \tau_{k_a}^i \leq m_1 h_{\min}$, where $h_{\min} \leq \tau_h \leq h_{\max}$. Otherwise, the transmission delay $t^k_{m_1}$ of packet $u_{k-m_1}$ will exceed the upper bound of the transmission delay, i.e. $t^k_{m_1} > h_{k-1} + h_{k-2} + \ldots + h_{k-m_1} \geq m_{1}h_{\max}$.

We also know if packet $u_{k-m_1}$ is lost and consecutive $m_2 - 1$ packets sent before it are also dropped during the transmission, then packet $u_{k-m_1-m_2}$ (namely $u_{k-s}$) will be used to act on the plant. In this paper, a zero-order-holder (ZOH) is used in the NCS and the newest control signal is always used to control the plant. Thus, the control input to the plant is kept constant during the sampling interval $[t_k, t_{k+1}]$, and there are $s + 1$ cases. As a consequence, a sequence of packets $u_{k-s}, u_{k-s+1}, \ldots, u_k$ becomes the research object of analysing packet disordering.

### 2.1 Description of packet disordering

In this paper, packet disordering is characterised as follows:

i. It is well known that the corresponding expected arrival sequence numbers (EASN) are $1, 2, \ldots, s + 1$ for the packets $u_{k-s}, u_{k-s+1}, \ldots, u_k$. Then, it is easily obtained that the EASN of packet $u_{k-s}$ is $s + 1 - i \ (i = 0, 1, \ldots, s)$.

ii. When the packet $u_{k-s}$ arrives at the actuator, a receive_index $R_k(i)$ ($R_k(i) \in \{1, 2, \ldots, s + 1\}$) is assigned to it. The value of receive_index assigned to each received packet is determined by the real arrival order of the packet. For example, if the packet $u_{k-s}$ is the second to arrive at the actuator for the packets $u_{k-s}, u_{k-s+1}, \ldots, u_k$, then its receive_index is 2. Without loss of generality, we assume that the packets not appeared or lost before and inclusive of the time instant $t_k$ arrive at the actuator in order after time instant $t_k$. Moreover, if $p \ (p < m_2)$ and it is some positive integer) packets sent before the $u_{k-s}$ are dropped, then its receive_index is $p$ more than the real value.

iii. $d_k(s + 1 - i) = R_k(i) - (s + 1 - i)$ denotes the displacement value of packet $u_{k-i}$ ($i = 0, 1, \ldots, s$). For the packet $u_{k-s}$ arriving at the actuator before and inclusive of the time instant $t_k$, if $d_k(s + 1 - i) \neq 0$, then a ‘disordering event’ has occurred during transmission. Packet $u_{k-s}$ is late if $d_k(s + 1 - i) > 0$, early if $d_k(s + 1 - i) < 0$, and in order if $d_k(s + 1 - i) = 0$ (see [24, 25]).

Here, disordering percentage (DP) is defined below to report the proportion of disordered packets.

$$\text{DP}_k = \frac{n_{d,k}}{n_k} \times 100\% \quad (11)$$

### Table 1 Example in Fig. 3

<table>
<thead>
<tr>
<th>$x_k$</th>
<th>$x_{k-5}$</th>
<th>$x_{k-4}$</th>
<th>$x_{k-3}$</th>
<th>$x_{k-2}$</th>
<th>$x_{k-1}$</th>
<th>$u_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EASN</td>
<td>1 2 3 4 5</td>
<td>1 2 3 4 5</td>
<td>1 2 3 4 5</td>
<td>1 2 3 4 5</td>
<td>1 2 3 4 5</td>
<td></td>
</tr>
<tr>
<td>assigned receive_index values</td>
<td>2 1 6 7 8</td>
<td>2 1 6 7 8</td>
<td>2 1 6 7 8</td>
<td>2 1 6 7 8</td>
<td>2 1 6 7 8</td>
<td></td>
</tr>
<tr>
<td>displacement values</td>
<td>$d_k(6-i)$</td>
<td>$i = 0, 1, \ldots, 5$</td>
<td>$i = 0, 1, \ldots, 5$</td>
<td>$i = 0, 1, \ldots, 5$</td>
<td>$i = 0, 1, \ldots, 5$</td>
<td></td>
</tr>
</tbody>
</table>

where $n_{d,k}$ and $n_k$ denote the number of packet disordering and the focused sequence size in the time interval $[t_0, t_k]$, respectively. It is well known that $0 \leq \text{DP}_k \leq 1$.

To further understand packet disordering and disordering percentage, Table 1 lists the EASN, receive_index values and displacement values for the sequence in the example shown in Fig. 3. In this example, set $m_1 = 3$ and $m_2 = 2$, and thus $s = 5$. On the basis of the displacement values of packets listed in Table 1, we know that the packet $u_{k-5}$ is displaced by one unit from its position, the packet $u_{k-4}$ is early by one position, namely packets $u_{k-5}$ and $u_{k-4}$ arrived out of order. The packet $u_{k-1}$ is in order. We perceived both packets $u_{k-3}$ and $u_{k-2}$ as lost indicating $\text{DP}_k = 2/5$ during the time interval $[t_{k-5}, t_k]$.

### Remark 1

From Fig. 3, we can clearly realise that packet disordering is different from packet loss. Packets $u_{k-3}$ and $u_{k-2}$ are perceived as lost, so from the communication point of view, the packet loss rate is 2/5. However, from the perspective of the closed-loop NCS, coming late $u_{k-5}$ is discarded, and thus the packet loss rate is 3/5. So, packet disordering results in the decrease of reliability of an NCS.

### 2.2 Modeling of an NCS

In terms of displacement values of packets, the following two operators are defined

$$\delta(d_k(s + 1 - i)) = \begin{cases} 1 & d_k(s + 1 - i) \leq 0 \\ 0 & d_k(s + 1 - i) > 0 \end{cases} \quad (12)$$

$$\theta_k(l) = \prod_{j=0}^{l-1} (1 - \delta(d_k(s + 1 - j))) \delta(d_k(s + 1 - l)) \quad (13)$$

where $\prod_{j=0}^{l-1} (1 - \delta(d_k(s + 1 - j))) = 1$.

![Fig. 3 Illustration of signals transmitting with packet disordering](image-url)
Remark 2: It is worth noting that $\theta_k(l) = 1$ or 0, and $\sum_{l=0}^{t} \theta_k(l) = 1$. $\theta_k(l) = 1$ or $\theta_k(l) = 0$ depends on the displacement values of packets. By defining $\theta_k(l+1-1)$ and $\theta_k(l)$ operators, the newest control input $(u_k(t_k+l) = 0, 1, \ldots, s)$ used for updating the plant during time interval $[t_k, t_{k+1})$ can be determined by $\sum_{l=0}^{t} \theta_k(l)u_k(t_k+l)$.

Thus, (5) is written as

$$x_{k+1} = \bar{A}_kx_k + \bar{B}_k\sum_{l=0}^{t} \theta_k(l)u_k(t_k+l)$$
$$x(t_0) = x_0$$

(14)

For the example in Fig. 3, we can easily calculate $\theta_k(0) = 0$, $\theta_k(1) = 1$, $\theta_k(2) = 0$, $\theta_k(3) = 0$, $\theta_k(4) = 0$ and $\theta_k(5) = 0$ according to (12) and (13). Thus, the newest control input is $u_{k-1}$.

Let $z_k = [x_k^T u_{k-1}^T \ldots u_{k-t-1}^T]^T$, then (14) is written as

$$z_{k+1} = G_kz_k + H_ku_k$$
$$z(t_0) = z_0$$

(15)

where

$$G_k = \begin{bmatrix} A_k & \bar{B}_k\theta_k(1) & \ldots & \bar{B}_k\theta_k(s-1) & \bar{B}_k\theta_k(s) \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{bmatrix}$$

$$H_k = \begin{bmatrix} \bar{B}_k\theta_k(0) \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3 LQR problem of the NCS

The objective of this paper is to find a suboptimal solution to the LQR problem of NCS (15) with performance index (4). For achieving this purpose, we intend to convert this optimisation problem into the standard form of LQR problem.

Proposition 1: Performance index (4) can be expressed as the following form

$$J_N = \frac{1}{2} \sum_{k=0}^{N} z_k^T \bar{P}_k z_k + \bar{u}_k^T R_k \bar{u}_k + z_{N+1}^T \bar{Q}_{N+1} z_{N+1}$$

(16)

where $\bar{u}_k$ is called a virtual controller. $\bar{P}_k = \hat{Q}_k - 2 \bar{W}_k R_k^{-1} \hat{W}_k^T$, $\hat{Q}_k = \text{diag}(Q_{k,0}, \ldots, 0)$, $\bar{Q}_{N+1} = \text{diag}(Q_{N+1,0}, \ldots, 0)$, $\bar{W}_k = \bar{W}^T_k$ and $e = [\bar{W}_k \ldots 0]$.

Proof: Equation (4) is first rewritten as

$$J_N = \frac{1}{2} \sum_{k=0}^{N} \tilde{z}_k^T \tilde{Q}_k \tilde{z}_k + \tilde{u}_k^T R_k \tilde{u}_k + \tilde{z}_{N+1}^T \tilde{Q}_{N+1} \tilde{z}_{N+1}$$

(17)

and then, we introduce a virtual controller

$$\tilde{u}_k = u_k + R_k^{-1} \bar{W}_k^T \tilde{z}_k$$

(18)

Thus, $u_k = \bar{u}_k - R_k^{-1} \bar{W}_k^T \tilde{z}_k$, and substitute it into (17), so it follows

$$J_N = \frac{1}{2} \sum_{k=0}^{N} \tilde{z}_k^T \tilde{Q}_k \tilde{z}_k + \bar{u}_k^T R_k \bar{u}_k + \bar{z}_{N+1}^T \bar{Q}_{N+1} \bar{z}_{N+1}$$

(19)

Finally, rewriting (19) yields (16). □

Proposition 2: NCS (15) can be written as

$$z_{k+1} = \Phi_k z_k + H_k \bar{u}_k$$
$$z(t_0) = z_0$$

(20)

where $\Phi_k = G_k - H_k R_k^{-1} \bar{W}_k^T$

Proof: Substituting $u_k = \bar{u}_k - R_k^{-1} \bar{W}_k^T z_k$ into (15) easily yields (20).

Problem 1: For NCS (20) and cost function (16), find a sampling period sequence $H^*(h_0^*, h_1^*, \ldots, h_N^*)$ and a control sequence $U^*(u_0^*, u_1^*, \ldots, u_N^*)$

such that cost function (16) is minimised, i.e.

$$\min_{h_0^*, h_1^*, \ldots, h_N^*} \sum_{k=0}^{N} \tilde{z}_k^T \tilde{Q}_k \tilde{z}_k + \tilde{u}_k^T R_k \tilde{u}_k + \tilde{z}_{N+1}^T \tilde{Q}_{N+1} \tilde{z}_{N+1}$$

s.t. $z_{k+1} = \Phi_k z_k + H_k \bar{u}_k$
$$z(0) = z_0$$

(21)

Remark 1: Obviously, (21) is a standard LQR problem. Once the virtual controller $\bar{u}_k$ is designed, the control input $u_k$ can be determined in terms of (18).

Remark 4: From (21), one can clearly see that the optimal virtual controller $\bar{u}_k$ depends on $\tilde{P}_k$, $\Phi_k$ and $H_k$, which are all the functions of sampling period $h_k$ and $\theta_k(l)$ ($l = 0, 1, \ldots, s$), and $\theta_k(l)$ is determined by the displacement values of packets. Hence, the optimal virtual controller $\bar{u}_k$ depends on sampling period and packet disordering.

4 Solution to the LQR problem of the NCS

This section is devoted to solving Problem 1 for the NCS with packet disordering. First, for given sampling period and parameters of the network, a method used to solve the standard LQR of the NCS is presented. And then, a novel sampling period selection (SPS) strategy is investigated based on disordering percentage. Finally, a sampling and control algorithm is proposed to find a suboptimal solution to Problem 1.

4.1 Solving the LQR problem when knowing system dynamics

Theorem 1: For given sampling period $h_k$ and parameters of the network $\theta_k(l)$ ($l = 0, 1, \ldots, s$), the suboptimal controller $u_k^*$ that
minimises (16) which is subject to NCS (20) can be calculated by
\[ u^*_k = -(S_k + R_k^{-1} \hat{W}_k^T) z_k \] (22)
and the optimal cost is
\[ J^* = \frac{1}{2} z_0^T P_0 z_0 \] (23)
with
\[ S_k = (R_k + H_k^T P_{k+1} H_k)^{-1} H_k^T P_{k+1} \Phi_k \] (24)
\[ P_k = \hat{F}_k + S_k^T R_k S_k + (\Phi_k - H_k S_k) P_{k+1} (\Phi_k - H_k S_k) \]
(25)
\[ P_{N+1} = \hat{Q}_{N+1} \]
(26)

**Proof:** We write a value function associated with (16) as
\[ V(z_k) = \min_a \left[ \frac{1}{2} \sum_{i=k}^{N} z_i^T \hat{F}_i z_i + u_i^T R_i u_i \right] \] (27)
A difference equation equivalent to this finite horizon sum is given by
\[ V(z_k) = \min_a \left[ \frac{1}{2} z_k^T \hat{F}_k z_k + u_k^T R_k u_k \right] + \frac{1}{2} \sum_{i=k+1}^{N} z_i^T \hat{F}_i z_i + u_i^T R_i u_i \] (28)
Actually, (28) is the Bellman equation for Problem 1. Since \( J_N \) is quadratic in the state, we assume
\[ V(z_k) = \frac{1}{2} z_k^T P_k z_k \] (29)
where matrix \( P_k \geq 0 \). We have
\[ 2V(z_k) = z_k^T \hat{F}_k z_k + u_k^T R_k u_k + z_{k+1}^T P_{k+1} z_{k+1} \] (30)
Furthermore, we have
\[ z_k^T P_k z_k = z_k^T \hat{F}_k z_k + u_k^T R_k u_k + z_{k+1}^T P_{k+1} z_{k+1} \] (31)
Substituting state equation (20) into (31) yields Lyapunov equation (25). The following is Hamilton function
\[ H(z_k, \hat{u}_k) = z_k^T \hat{F}_k z_k + u_k^T R_k u_k + z_{k+1}^T P_{k+1} z_{k+1} - z_k^T P_k z_k \]
\[ = z_k^T \hat{F}_k z_k + u_k^T R_k u_k + \Phi_k^T z_{k+1} (H_k \hat{u}_k) + (\Phi_k z_k + H_k \hat{u}_k) P_{k+1} (\Phi_k z_{k+1} + H_k \hat{u}_{k+1}) - z_k^T P_k z_k \] (32)
A necessary condition for optimality is given below
\[ \frac{\partial H(z_k, \hat{u}_k)}{\partial \hat{u}_k} = 2R_k \hat{u}_k + 2H_k^T P_{k+1} H_k \hat{u}_k + 2H_k^T P_{k+1} \Phi_k z_k = 0 \] (33)
Solving this yields the optimal control
\[ \hat{u}_k = -(R_k + H_k^T P_{k+1} H_k)^{-1} H_k^T P_{k+1} \Phi_k z_k \] (34)
By (18), (24) and (34), we can obtain (22). Referring to (27), the optimal value (23) can be obtained in terms of (29).

**Remark 5:** If sampling period \( h_k \) and parameter \( \theta_k(l) \) are known, then the dynamics of NCS (20) is known. In this case, Theorem 1 presents an analytical solution to Problem 1. The solution \( u^*_k \) is calculated in terms of received state \( x_k \) and previous control input \( u^*_j, j = k, k+1, \ldots, \) that was used to update the plant. If \( P_{k+1}, h_k \) and \( \theta_k(l) \) are given, then \( S_k, P_k \) and \( P_{N+1} \) can be calculated in terms of Riccati difference recursion equations (24) and (25), but which requires the future sampling period \( h_{k+1} \) and parameter \( \theta_{k+1}(l) \) to be known.

We note that the sampling periods are assumed to be known in Theorem 1. Thus, packets will be sent at the specific time instant over the network, and meanwhile the displacement values of packets and disordering percentage can be calculated at the \( t_k \) (\( k = 1, 2, \ldots, N \)) time instant. Then we can calculate parameters \( \theta_k(l) \) (\( k = 1, 2, \ldots, N; l = 0, 1, \ldots, s \)) until the \( t_N \) time instant and store them in the controller for being used to calculate matrices \( S_k \) and \( P_k \), and further to calculate the optimal control law \( u_k \). That means that the parameters \( \theta_k(l) \) (\( l = N - k, N - k + 1, \ldots, 0; k = 1, \ldots, N \)) have to be available at the \( t_k \) time instant.

### 4.2 SPS scheme

The focus in this paper is to find a suboptimal solution to optimisation Problem 1 by determining a sampling period sequence and designing a control input sequence. Note that Theorem 1 has presented a method of designing optimal controller, but it requires sampling period \( h_k \) and parameters \( \theta_k(l) \) (\( l = 0, 1, \ldots, s \)) to be known. In this subsection, the relationship among sampling period, packet disordering and quadratic performance index is analysed and verified. Based on it, an SPS scheme is proposed.

Gharai et al. [22] provided an interesting research that illustrated and confirmed the correlation between packet disordering and data rate. i.e. at higher data rate, packets with fixed packet size were more prone to packet disordering for UDP flows. In general, the packets from the same source go through the same link, but part of packets will be rescheduled to other parallel links when sending rate exceed the bottleneck of some router nodes or the capacity of some links, which would result in some packets ‘catching-up’ with those sent earlier, but queued on a parallel link due to short sending time interval (see path scheduling algorithm [26]), since for any two packets continuously sent from the same source, packet disordering occurs if and only if \( \Delta t = t_{k+1} - t_k > \beta_k \) holds [5]. In this paper, we choose UDP flow and assume that the sampled data must be sent by the sensor. So the sampling rate is approximately equivalent to data rate. Actually, the relation \( data\ rate = (2 * packet\ size) * sampling\ rate \) holds for a single closed-loop NCS, where \( packet\ size \) denotes the size of packets.

To further clarify the correlation between sampling rate and packet disordering, we construct a network topology by matlab software as shown in Fig. 4a, where the links from router 1, router 2 and router 3 to the receiver are, respectively, called Link 1, Link 2 and Link 3. Table 2 lists the length, transmission delay and bottleneck in the above links. Set \( packet\ size = 500 \) bits and utilise the path scheduling scheme shown in Table 3. In Fig. 4a, the lengths of Link 1, Link 2 and Link 3 are 44.7782, 43.7472 and 39.0218 m, respectively. Thus different transmission delay has occurred in the three links. This means that packets sent late are likely to arrive at the receiver earlier if they travel a short path to the receiver leading to out-of-order phenomenon. The transmission delays in the three links are generated by using Monte Carlo method.

<table>
<thead>
<tr>
<th>Table 2 Allocation of links</th>
</tr>
</thead>
<tbody>
<tr>
<td>Link 1</td>
</tr>
<tr>
<td>bottleneck, kbps</td>
</tr>
<tr>
<td>length, m</td>
</tr>
<tr>
<td>delay, s</td>
</tr>
</tbody>
</table>
First, the fixed sampling scheme is executed (see [4, 5, 14, 24]) in order to analyse the relationship between sampling period and packet disordering. Fig. 4b shows the disordering percentages when choosing fixed sampling periods 0.11, 0.08, 0.07 and 0.06 s. From Fig. 4b, the small sampling period causes that data rate exceeds the prescribed bottleneck of the links resulting in the packets from the same source being transmitted by the different links, thereby increasing packet disordering.

Second, to visualise the impact of sampling rate and packet disordering on the control performance index (4), the following second-order system is considered

\[
\dot{x}(t) = \begin{bmatrix} 0.2607 & -0.1266 \\ -0.4955 & 0.5046 \end{bmatrix} x(t) + \begin{bmatrix} 0.0255 \\ -0.2202 \end{bmatrix} u(t)
\]

The sampling period is set to 0.115, 0.11, 0.08, 0.07 and 0.06 s, respectively, then displacement values of packets, disordering percentage and parameters \(h_i\) \((i = 1, 2, \ldots, N; i = 0, 1, \ldots, s)\) can be calculated by checking the network when adopting each given sampling period. Thus the control law that minimises the cost (16) is derived by Theorem 1. Table 4 lists the cost values under the aforementioned sampling periods. From Table 4, one can find that if higher sampling rate is set, packet disordering becomes heavier. Moreover, as expected, one can also find that the control performance is not a monotonous function of sampling period, which is consistent with the results shown in Fig. 2 [10]. This means that sampling rate cannot be too fast, nor it is too slow, such that packet disordering is limited within a proper range and better control performance can be achieved.

Finally, based on the aforementioned analysis, we intend to propose an SPS scheme in order to control disordering percentage into a specific range. We define three levels of network operation including a poor network condition band, an acceptable network condition band and an excessively conservative network operation based on disordering percentage. When disordering percentage exceeds the upper bound \(DP_{UB}\), communication is entered into the poor network condition band due to excessive traffic, an upper and lower bounds of disordering percentage \(DP_k\) \((DP_{UB} \text{ and } DP_{LB}, \text{ respectively})\) is used to bound an acceptable network band, and an excessively conservative network operation is defined as the case of low percentage of packet disordering and even no out-of-order packets.

Set \(\Omega_1 = \{k|DP_k \in (DP_{UB}, 1]\}, \Omega_2 = \{k|DP_k \in [DP_{LB}, DP_{UB}]\}, \Omega_3 = \{k|DP_k \in [0, DP_{LB}]\}\) and define a variable \(\beta_k\) as

\[
\beta_k = \begin{cases} 1 & k \in \Omega_1 \\ 0 & k \in \Omega_2 \\ -1 & k \in \Omega_3 \end{cases}
\]

It is clear that \(\Omega_1 \cup \Omega_2 \cup \Omega_3 = Z, \Omega_i \cap \Omega_j = \phi \ (i \neq j, i, j = 1, 2, 3)\), where \(Z\) and \(\phi\) denote set of non-negative integer and empty sets, respectively. An SPS scheme is given as follows

\[
h_{k+1} = h_k + \beta_k \Delta h
\]

where \(\Delta h\) is a positive step-size and \(0 \leq \Delta h \leq 1\).

Remark 6: If the disordering percentage \(DP_k\) exceeds the upper threshold \(DP_{UB}\), moving into the poor network condition, \(h_k\) is increased by \(\Delta h\) to take \(DP_k\) back into the acceptable region. When \(DP_k\) decreases into the excessively conservative region below

---

**Table 3** Path scheduling

<table>
<thead>
<tr>
<th>Rate, kbps</th>
<th>Choose Link 1, %</th>
<th>Choose Link 2, %</th>
<th>Choose Link 3, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 9</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16–9</td>
<td>50</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>≥ 16</td>
<td>40</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

---

**Table 4** Comparisons of performance

<table>
<thead>
<tr>
<th>Sampling periods</th>
<th>Average of (DP_k)</th>
<th>Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.115</td>
<td>0</td>
<td>7.8880 × 10^6</td>
</tr>
<tr>
<td>0.11</td>
<td>0</td>
<td>3.7565 × 10^5</td>
</tr>
<tr>
<td>0.08</td>
<td>0.11</td>
<td>226.0045</td>
</tr>
<tr>
<td>0.07</td>
<td>0.2371</td>
<td>1.0605 × 10^5</td>
</tr>
<tr>
<td>0.06</td>
<td>0.2756</td>
<td>1.3963 × 10^5</td>
</tr>
</tbody>
</table>

---

**Fig. 4** Network topology by matlab software

a Topology of communication network

b Disordering percentages under the different sampling periods
Comparisons of performance

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>...</th>
<th>$d_{n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>0</td>
<td>$f_{12}$</td>
<td>$f_{13}$</td>
<td>...</td>
<td>$f_{1(n+1)}$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>0</td>
<td>0</td>
<td>$f_{23}$</td>
<td>...</td>
<td>$f_{2(n+1)}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d_n$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>$f_{n(n+1)}$</td>
</tr>
<tr>
<td>$d_{n+1}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
</tbody>
</table>

$D_P^{LB}$, we shall decrease the sampling period by $\Delta h$ to fully utilise the network band and avoid unstabilisability of NCSs due to less transferred information.

4.3 Suboptimal sampling and control strategy

From (36) and (37), one can find that we try to limit disordering percentage within the range of $[D_P^{LB}, D_P^{UB}]$ by increasing or decreasing the sampling rate. From Table 4, one can find that if $[D_P^{LB}, D_P^{UB}]$ can be appropriately chosen, i.e. disordering percentage is limited within a proper range, then a suboptimal cost can be obtained by SPS scheme (37) and Theorem 1. Hence, we propose Algorithm 1 (see Fig. 5) to solve Problem 1. We assume that $0 \leq \gamma_1 \leq D_P^{LB} < D_P^{UB} < \gamma_2 \leq 1$, and $\gamma_1$ and $\gamma_2$ can be determined by testing the real network.

Remark 7: By executing Algorithm 1 (Fig. 5), we can in fact obtain Table 5, where the first row and the first column, respectively, denote the possible values of $D_P^{LB}$ and $D_P^{UB}$, and the interior of the form lists the corresponding cost values $f_{ij}$, ($i, j = 1, 2, \ldots, n + 1$) calculated in terms of (23). By comparing all values of $f_{ij}$, we can find the minimum cost $f_{\text{min}}$, thus a suboptimal solution of Problem 1, i.e. suboptimal sampling period sequence $h_k^*$ and control input sequence $u_k^*$ ($k = 0, 1, \ldots, N$) is derived since SPS (37) is heuristic. In addition, note that $f_{ij} = 0$ if $i < j$ in Table 5 since $D_P^{LB} < D_P^{UB}$. In this paper, we do not take the special case $DP_{LB} = DP_{UB}$ into account, because frequently varying sampling period might destabilise the NCS since disordering percentage is not constant due to dynamical networks.

Remark 8: In step 3 of Algorithm 1 (Fig. 5), we first adjust sampling period in terms of SPS scheme (37) when $DP_{LB}$ and $DP_{UB}$ have been chosen. It should be pointed out that sampling period is changed according to disordering percentage and meanwhile we calculate parameters $\theta_k(l)$ $(l = 0, 1, \ldots, s)$ in online real-time approach by checking the network, which means that once sampling period is chosen, packets will be sent at the specific time instant over the network, such that displacement values of packets and disordering percentage can be calculated for the focused network, further parameters $\theta_k(l)$ can be determined. In this way, all sampling periods $h_k$ and parameters $\theta_k(l)$ ($k = 1, 2, \ldots, N$) are calculated and stored in the controller to calculate matrices $S_k$ and $P_k$ as stated in Remark 5, further to calculate $u_k^*$ by Theorem 1.

Remark 9: One can find that comparison between $f_{ij}$ and $f_{\text{min}}$ is made $(n(n + 1))/2$ times from Table 5. Actually, the bigger $n$ is, the more precise the computational results are but at the loss of increasing computational time.

5 Simulation results

In this section, a numerical example and a networked DC motor control system are given to verify the effectiveness of the proposed method.

5.1 Numerical example

Consider the example in Section 4, Fig. 6 shows the sampling period variation corresponding to disordering percentage when performing SPS scheme (37) under the different $D_P^{LB}$ and $D_P^{UB}$.

Algorithm 1:

1: Partition the interval $[\gamma_1, \gamma_2]$ into $n$ ($n$ is a positive real integer) equal parts. Set $\gamma_1 = d_1 < d_2 < \cdots < d_{n+1} = \gamma_2$. Thus, it follows $d_{i+1} - d_i = \lambda = (\gamma_2 - \gamma_1)/n$.
2: Let $i = 1$, $j = 1$ and $f_{\text{min}} = 0$;
3: Let $j = j + 1$, and set $D_P^{LB} = d_i$ and $D_P^{UB} = d_j$;
4: Execute SPS scheme (37) in order to adjust sampling period $h_k$

$(k = 0, 1, \cdots, N)$, based on which packets are sampled and sent to the network, and then compute parameters $\theta_k(l)$ $(l = 0, 1, \cdots, s)$;
5: Calculate $S_k$ and $P_k$ by Theorem 4.1 when knowing sampling period $h_k$

$(k = 1, 2, \cdots, N)$ and parameters $\theta_k(l)$ $(l = 0, 1, \cdots, s)$. Calculate cost $J^*$

according to (23) and set $f_{ij} = J^*$;
6: If $f_{ij} < f_{\text{min}}$, then $f_{\text{min}} = f_{ij}$, $D_P^{LB} = d_i$, $D_P^{UB} = d_j$, $h_k = h_k$ and $u_k^* = u_k$ $(k = 0, 1, \cdots, N)$. Otherwise, go to Step 6;
7: If $j < n + 1$, go to Step 2. Otherwise, let $j = i + 1$, and go to Step 7;
8: Output $h_k^*$, $u_k^*$ and $f_{\text{min}}$. 

Fig. 5 Searching the suboptimal solution to Problem 1
Fig. 6  Sampling period variation corresponding to disordering percentage under the different $DP_{LB}$ and $DP_{UB}$

$a$ $DP_k \in [0.2, 0.3]$

$b$ $DP_k \in [0.1, 0.2]$

$c$ $DP_k \in [0.05, 0.15]$

$d$ $DP_k \in [0.02, 0.05]$

$e$ $DP_k \in [0.02, 0.03]$

$f$ $DP_k \in [0.01, 0.02]$

Table 6  Comparisons of performance

<table>
<thead>
<tr>
<th>$DP_{LB}$</th>
<th>$DP_{UB}$</th>
<th>Costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>877.6266</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2</td>
<td>586.7316</td>
</tr>
<tr>
<td>0.05</td>
<td>0.15</td>
<td>584.6686</td>
</tr>
<tr>
<td>0.02</td>
<td>0.05</td>
<td>523.3194</td>
</tr>
<tr>
<td>0.02</td>
<td>0.03</td>
<td>586.6924</td>
</tr>
<tr>
<td>0.01</td>
<td>0.02</td>
<td>883.0201</td>
</tr>
</tbody>
</table>

Table 6 lists the corresponding suboptimal costs. In this example, we set $h_{\text{min}} = 0.06$ s and $h_{\text{max}} = 0.11$ s. Compared with the costs listed in Table 4 in which the fixed sampling period [4, 5, 14, 24] is used for the NCS, the better control performance is obtained by using SPS scheme (37). Even though when the sampling period is 0.08 s, the cost value is smallest, but it is quite hard to find it only by checking sampling period. By Algorithm 1 (Fig. 5),

Table 7  Allocation of links

<table>
<thead>
<tr>
<th>Link 1</th>
<th>Link 2</th>
<th>Link 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>bottleneck, kbps</td>
<td>326</td>
<td>390</td>
</tr>
<tr>
<td>delay, s</td>
<td>(0.001,0.002)</td>
<td>(0.0005,0.001)</td>
</tr>
</tbody>
</table>

5.2 Application of DC motor NCS

The following gives the dynamics of DC motor

$$\dot{i}_a = -\frac{R}{L}i_a - \frac{K_b}{L}w + \frac{1}{L}u$$

$$\dot{\omega} = \frac{K}{J}i_a - \frac{B}{J}\omega$$

(38)

where $\dot{i}_a = di_a/dt$, $\dot{\omega} = d\omega/dt$ and $i_a$, $\omega$, $R$, $L$, $K_b$, $K$, $J$, $B$ and $u$ are armature winding current, rotor angular speed, armature

Fig. 7  Optimal cost searching

© The Institution of Engineering and Technology 2016
Letting \( x = [i_a \ \omega]^{\top} \) yields the expression below

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

(39)

where \( y(t) \) denotes rotor angular speed, and

\[
A = \begin{bmatrix} R & K_b \\ L & -L \\ K & -B \\ J & -J \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ L \end{bmatrix}, \\
C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0
\]

Here, we choose \( R = 2.6 \, \Omega \), \( L = 2 \times 10^{-3} \, H \), \( J = 1.2 \, kg \cdot m^2 \), \( K_b = 0.77769 \, V \cdot m \cdot s \), \( K = 0.7 \, N \cdot m \cdot s \) and \( B = 0.01 \, N \cdot m \cdot s \).

DC motor (38) exchanges data with the controller over the simulated network shown in Fig. 4a, where the bottleneck and transmission delay in the links are listed in Table 7. Table 8 lists the path scheduling scheme. Under the initial condition \( x_0 = [1 \quad 1]^T \), by Algorithm 1 (Fig. 5), the suboptimal cost value 1.0379 is obtained when \( DP_k \in [0.16, 0.18] \). Fig. 7 shows how to search the suboptimal cost. Here, the simulation results cannot be compared with the ones obtained by [10, 11], since they did not consider packet disordering. Figs. 8 and 9 show the interactive process between sampling period and packet disordering and the state response of NCS when \( DP_k \in [0.16, 0.18] \), respectively.

Table 8 Path scheduling

<table>
<thead>
<tr>
<th>Rate, kbps</th>
<th>Choose Link 1, %</th>
<th>Choose Link 2, %</th>
<th>Choose Link 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate &lt; 326</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>390 &gt; rate ≥ 326</td>
<td>50</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>406 &gt; rate ≥ 390</td>
<td>40</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

6 Conclusions
In this paper, we formulate LQR problem to find a suboptimal sampling and control strategy for a NCS with packet disordering, transmission delay and packet loss. By defining the displacement values of packets, a discrete-time system model of the NCS is put forward. Based on it, the optimal controller gains are derived when knowing sampling period and QoS of the network. An SPS scheme is proposed based on the relationship between packet disordering and sampling period. Furthermore, a numerical algorithm is proposed by integrating the SPS scheme and LQR problem in order to provide a suboptimal sampling period sequence and control input sequence for the LQR problem of the NCS. The simulation results demonstrate the effectiveness of the developed approach.

7 Acknowledgments
The authors thank Prof Xie Lihua at Nanyang Technological University, Singapore for his valuable suggestions on this
paper and also acknowledge the National Natural Science Foundation of China under grants 61104093, 61525302, 61333012, 61304028, 61590022, 61503257 and the Program for Liao Ning Province Distinguished Scholars in University of China under grant LJQ2015088.

8 References