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Asymptotically necessary and sufficient stability conditions for discrete-time Takagi–Sugeno model: Extended applications of Polya's theorem and homogeneous polynomials

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Abstract

The stability and stabilization conditions of the nonlinear system in Takagi–Sugeno's form are considered. The homogeneously polynomially nonquadratic (HPNQ) Lyapunov functions and homogeneously polynomially parameterized (HPP) state feedback laws are adopted. By generalizing the procedure based on the Polya's theorem, the asymptotically necessary and sufficient (ANS) stability and stabilization conditions in the case of HPNQ Lyapunov functions and HPP control laws are reformulated. The major contribution of this paper is to give the parallel results using the multiple indices, so that the slack matrices can be extensively utilized to improve the numerical efficiency. The effectiveness of the results is illustrated by the numerical examples.

1. Introduction

A well-known approach to the fuzzy control is based on the so-called Takagi–Sugeno (T–S) models [26]. The stability and stabilization of nonlinear control systems based on T–S models have attracted great attentions and most of the conditions are expressed in linear matrix inequalities (LMIs) which can be effectively solved via the commercially available software [12]. The earlier results on stability analysis and stabilization are based on the common quadratic Lyapunov functions [28] and the parallel distributed compensation (PDC) law [30]. It has been

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observed that the stability and stabilization results based on the common Lyapunov functions tend to be conservative in general, and quite often the solution might not exist at all, especially for highly complex nonlinear systems. Considerable efforts have been devoted to developing less conservative approaches to stability analysis and stabilization for the past years, and many significant results have been obtained.

One of the successful approaches to less conservative stability analysis and stabilization design is based on the so-called piecewise quadratic Lyapunov functions [1,10,11,15]. Another approach to reducing the conservatism in stability analysis or stabilization results is to adopt fuzzy Lyapunov functions [27]. In other contexts, fuzzy Lyapunov functions are also called parameter-dependent Lyapunov functions [2,5], or nonquadratic Lyapunov functions [14], or poly-quadratic Lyapunov functions [3]. It is noted that fuzzy Lyapunov functions are more effective for discrete-time systems than for continuous-time systems. Other nonquadratic Lyapunov functions different from fuzzy Lyapunov functions are also proposed in [14]. The authors in [6] utilized extended nonquadratic Lyapunov functions to obtain further improved results.

In addition to the above-mentioned approaches, there are also some other approaches to reducing the conservatism. One of the approaches is to introduce some extra slack variables in LMI [14,16,18,19,25]. Other approaches include adopting the improved form of feedback control laws [13,14] and utilizing higher-ordered fuzzy summations [4,9,29]. It has been shown that all these techniques are able to reduce the conservatism of the resulting stability or stabilization results to certain extent.

More recently, the Polya's theorem has been used in [20,25] for stability analysis and stabilization of T–S fuzzy systems, where asymptotically necessary and sufficient (ANS) stability or stabilization conditions in the sense of common quadratic Lyapunov functions have been obtained. In this paper, as a special case, it is shown that the ANS stability conditions in the sense of the extended nonquadratic Lyapunov function in [6] can be obtained. However, the ANS stability conditions in the sense of the nonquadratic Lyapunov functions in [14] cannot be obtained. This represents an important merit of [6] over [14]. The work [25] includes [14] as a special case, but it cannot include [6]. The authors in [21] studied the stability of time-invariant linear system based on the Polya's theorem and homogeneously polynomially parameter-dependent (HPPD) Lyapunov functions, but not primarily for fuzzy control. The author in [8] extended the results of [21] to T–S fuzzy model and HPP non-parallel distributed compensation (HPP-non-PDC) law. With the increase of the complexity of the HPPD Lyapunov functions, any Lyapunov function continuous on the membership functions can be approximated. With the increase of the complexity of the HPP-non-PDC laws, any control law continuous on the membership functions can be approximated. Hence, Ding [8] has obtained the ANS stabilization conditions in the sense of any Lyapunov function and any control law which are continuous on the membership functions.

In [21], the ANS stability conditions in the sense of the membership function-dependent Lyapunov matrix have been obtained. In [20,25], the ANS stability conditions for the membership function-dependent model have been obtained. The work [8] generalizes the results in [20,21,25], i.e., [8] gives the ANS stability conditions for both the membership function-dependent model and the membership function-dependent Lyapunov matrix (or feedback control gain). In [17], the Lyapunov function and control law having the similar role as those in [8] have been adopted. However, [17] has not shown that it obtains any kind of ANS stability result. While the results have been proposed in [8], the work [31] claims to have obtained less conservative results. As a matter of fact, [31] has applied the Lyapunov function in [7], and obtained some sufficient conditions with respect to the specified Lyapunov function and control law. The results in [8] cannot be further improved with respect to the *ad hoc* studied issue. In [32], the authors have applied the results in [6,8] to handle the bounded disturbance and input/state physical constraints. In the present paper, based on the results in [7], the results parallel to [8] will be obtained. Different from [8], the present paper utilizes the multi-index as in [25], and the slack matrices which are generalization and improvement of [25].

This paper is organized as follows. In Section 2, the notations are introduced. In Section 3, HPNQ Lyapunov functions and HPP-non-PDC and HPPPDC laws are utilized to improve the existing results. Section 4 generalizes the procedures in [25], i.e., utilizes Polya's theorem to obtain the ANS stability or stabilization conditions. Section 5 gives a numerical example and Section 6 concludes the paper.

Notations: The symbol (⁎) in a symmetric matrix denotes the transposed element in the symmetric position. The time-dependence of the variables is often omitted for brevity.

2. Notations for fuzzy summations

Consider a discrete-time T–S fuzzy model

$$x(t+1)=\sum_{i=1}^{r} h_i(z(t))(A_i x(t)+B_i u(t)), \tag{2.1}$$

where $u\in\Re^{n_u}$, $x\in\Re^{n_x}$ and $z\in\Re^{n_z}$ are the input, the measurable state and the measurable premise variable, respectively; $h_i(z(t))$ denotes the membership function, and it is assumed that $h_i(z(t))\geq 0$ and $\sum_{i=1}^{r} h_i(z(t))=1$.

The following notations will be adopted for simplicity:

$h_i=h_i(z(t))$, $h_i^+=h_i(z(t+1))$, $\sum_i=\sum_{i=1}^r$;
for a set of matrices X_i, $i=1,\ldots,r$, $X_z:=\sum_i h_i X_i$;
for a set of matrices $X_{ij}, i,j=1,\ldots,r$, $X_{zz}:=\sum_i\sum_j h_i h_j X_{ij}$, $X_{zz+}:=\sum_k\sum_l h_k^+ h_l^+ X_{kl}$.

The multi-index notations in [25] are properly adopted, mainly $\mathbb{I}_n=\{\mathbf{i}=(i_1,i_2,\ldots,i_n)|i_j=1,2,\ldots,r,\ \forall j=1,2,\ldots,n\}$, $\mathbb{I}_n^+=\{\mathbf{i}\in\mathbb{I}_n|i_k\leq i_{k+1},k=1,\ldots,n-1\}$. Given a multi-index $\mathbf{i}\in\mathbb{I}_n$, $\mathcal{P}(\mathbf{i})\subset\mathbb{I}_n$ denotes the set of permutations (with, possibly, repeated elements) of the multi-index $\mathbf{i}$. Besides, let $\mathcal{P}(\mathbf{i}\in\mathbb{I}_p,\mathbf{j}\in\mathbb{I}_{n-p}^+)$ being the subset of permutations of the multi-index $(\mathbf{i},\mathbf{j})$ for $\mathbf{i}\in\mathbb{I}_p,\mathbf{j}\in\mathbb{I}_{n-p}^+$. $\mathcal{P}(\mathbf{i}\in\mathbb{I}_p,\mathbf{j}\in\mathbb{I}_{n-p}^+)$ is different from $\mathcal{P}((\mathbf{i},\mathbf{j}))$ of [25]

$$\mathcal{P}(\mathbf{i}\in\mathbb{I}_p,\mathbf{j}\in\mathbb{I}_{n-p}^+)=\{(\mathbf{s},\mathbf{v})\in\mathcal{P}((\mathbf{i},\mathbf{j}))|\mathbf{s}\in\mathbb{I}_p,\ \mathbf{v}\in\mathbb{I}_{n-p}^+\}.$$

Similarly,

$$\mathcal{P}(\mathbf{i}\in\mathbb{I}_p^+,\mathbf{j}\in\mathbb{I}_{n-p})=\{(\mathbf{s},\mathbf{v})\in\mathcal{P}((\mathbf{i},\mathbf{j}))|\mathbf{s}\in\mathbb{I}_p^+,\ \mathbf{v}\in\mathbb{I}_{n-p}\}.$$

Example 1. Consider $i=1,\mathbf{j}=(1,1,2)$. Then, $\mathcal{P}((i,\mathbf{j}))=\{(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1)\}$ and $\mathcal{P}(i\in\mathbb{I}_1,\mathbf{j}\in\mathbb{I}_3^+)=\{(1,1,1,2),(2,1,1,1)\}$.

Let us also establish the equivalence to the notations in [8,20,21]. A homogeneous polynomial of degree p with respect to matrix X can be generally written as [21]

$$X_p(h)=\sum_{\mathbf{p}\in\mathcal{K}(p)} h^{\mathbf{p}}X_{\mathbf{p}},\quad h^{\mathbf{p}}=h_1^{p_1}h_2^{p_2}\cdots h_r^{p_r},\quad \mathbf{p}=p_1p_2\cdots p_r. \tag{2.2}$$

Here, $\mathcal{K}(p)$ is the set of r-tuples obtained as all possible combinations of nonnegative integers $p_i, i=1,\ldots,r$, such that $p_1+p_2+\cdots+p_r=p$. The number of elements in $\mathcal{K}(p)$ is $J(p)=(r+p-1)!/p!(r-1)!$. On the other hand, according to "Proposition 1" in [25]

$$\sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}X_{\mathbf{i}} = \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}} \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} X_{\mathbf{j}} = \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\tilde{X}_{\mathbf{i}}. \tag{2.3}$$

The number of elements in $\mathbb{I}_p^+$ is $NC_1=(r+p-1)!/p!(r-1)!$. Considering Eqs. (2.2) and (2.3), for each $\mathbf{p}$ in Eq. (2.2), there is a unique $\mathbf{i}$ in Eq. (2.3), such that $h_{\mathbf{i}}=h^{\mathbf{p}}$. Since $NC_1=J(p)$, we can choose $X_{\mathbf{p}}=\tilde{X}_{\mathbf{i}}$ for $h_{\mathbf{i}}=h^{\mathbf{p}}$. Then

$$X_p(h) = \sum_{\mathbf{p}\in\mathcal{K}(p)} h^{\mathbf{p}}X_{\mathbf{p}} = \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\tilde{X}_{\mathbf{i}}, \quad X_{\mathbf{p}} = \tilde{X}_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} X_{\mathbf{j}}. \tag{2.4}$$

Refer to $\mathcal{K}_\ell(p)$ and $\mathcal{C}^\ell(p)$ in [20]. $\mathcal{K}_\ell(p)$ is the ℓ-th r-tuple of $\mathcal{K}(p)$, $\ell=1,\ldots,J(p)$. For $h_{\mathbf{i}}=h^{\mathbf{p}}$, the number of elements in $\mathcal{P}(\mathbf{i})$ is $\mathcal{C}^\ell(p)|_{h_{\mathbf{i}}=h^{\mathbf{p}}}=p!/p_1!p_2!\cdots p_r!$, i.e., in Eq. (2.4), each $\tilde{X}_{\mathbf{i}}$ is a summation of $\mathcal{C}^\ell(p)|_{h_{\mathbf{i}}=h^{\mathbf{p}}}$ number of $X_{\mathbf{j}}$'s. In the following, for simplicity we denote

$$X_p(h) := \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\tilde{X}_{\mathbf{i}}, \quad \mathcal{C}_p(\mathbf{i}) := \mathcal{C}^\ell(p)|_{h_{\mathbf{i}}=h^{\mathbf{p}}}.$$

Example 2. Consider Eq. (2.4). For $p=3$ and $r=2$, $\mathcal{K}(p)=\{03,12,21,30\}$, $\mathbb{I}_p=\{111,112,121,122,211,212,221,222\}$ and $\mathbb{I}_p^+=\{111,112,122,222\}$. Then

$$\sum_{\mathbf{p}\in\mathcal{K}(p)} h^{\mathbf{p}}X_{\mathbf{p}} = h_2^3X_{03}+h_1h_2^2X_{12}+h_1^2h_2X_{21}+h_1^3X_{30},$$

$$\sum_{\mathbf{i}\in\mathbb{I}_p} h^{\mathbf{i}}X_{\mathbf{i}} = h_1^3X_{111}+h_1^2h_2(X_{112}+X_{121}+X_{211})+h_1h_2^2(X_{122}+X_{212}+X_{221})+h_2^3X_{222},$$

$$\sum_{\mathbf{i}\in\mathbb{I}_p^+} h^{\mathbf{i}}\tilde{X}_{\mathbf{i}} = h_1^3\tilde{X}_{111}+h_1^2h_2\tilde{X}_{112}+h_1h_2^2\tilde{X}_{122}+h_2^3\tilde{X}_{222}.$$

By choosing $X_{30}=\tilde{X}_{111}=X_{111}$, $X_{21}=\tilde{X}_{112}=X_{112}+X_{121}+X_{211}$, $X_{12}=\tilde{X}_{122}=X_{122}+X_{212}+X_{221}$ and $X_{03}=\tilde{X}_{222}=X_{222}$, Eq. (2.4) holds.

For some $\Upsilon_{\mathbf{i}}^{\mathbf{k}}$ (assumed symmetric, without loss of generality), the $(q\times p)$-dimensional fuzzy summation is denoted as $\Xi=\sum_{\mathbf{k}\in\mathbb{I}_q} h_{\mathbf{k}}^+ \sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}\Upsilon_{\mathbf{i}}^{\mathbf{k}}$ where, according to [25], $h_{\mathbf{i}}=\prod_{s=1}^p h_{i_s}=h_{i_1}h_{i_2}\cdots h_{i_p}$, $h_{\mathbf{k}}^+=\prod_{s=1}^q h_{k_s}^+=h_{k_1}^+h_{k_2}^+\cdots h_{k_q}^+$. Compared with $Q_{\mathbf{i}}$ in [25] which is only sub-indexed, here Υ is super- and sub-indexed. By applying "Proposition 1" in [25] for both the supper- and sub-indexes, we can obtain

$$\begin{aligned}\sum_{\mathbf{k}\in\mathbb{I}_q} h_{\mathbf{k}}^+ \sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}\Upsilon_{\mathbf{i}}^{\mathbf{k}} &= \sum_{\mathbf{k}\in\mathbb{I}_q} h_{\mathbf{k}}^+ \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}} \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} \Upsilon_{\mathbf{j}}^{\mathbf{k}} = \sum_{\mathbf{k}\in\mathbb{I}_q} h_{\mathbf{k}}^+ \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\check{\Upsilon}_{\mathbf{i}}^{\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\mathbb{I}_q^+} h_{\mathbf{k}}^+ \sum_{\mathbf{l}\in\mathcal{P}(\mathbf{k})} \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\check{\Upsilon}_{\mathbf{i}}^{\mathbf{l}} = \sum_{\mathbf{k}\in\mathbb{I}_q^+} h_{\mathbf{k}}^+ \sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}}\end{aligned} \tag{2.5}$$

where

$$\check{\Upsilon}_{\mathbf{i}}^{\mathbf{k}} = \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} \Upsilon_{\mathbf{j}}^{\mathbf{k}}, \quad \tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}} = \sum_{\mathbf{l}\in\mathcal{P}(\mathbf{k})} \check{\Upsilon}_{\mathbf{i}}^{\mathbf{l}} = \sum_{\mathbf{l}\in\mathcal{P}(\mathbf{k})} \sum_{\mathbf{j}\in\mathcal{P}(\mathbf{i})} \Upsilon_{\mathbf{j}}^{\mathbf{k}}.$$

In the literature, the condition for stability of a closed-loop fuzzy control system has been expressed, for some Υ_{ij}^{kl}, in the form

$$\Gamma = \sum_k \sum_l h_k^+ h_l^+ \sum_i \sum_j h_i h_j \Upsilon_{ij}^{kl} > 0. \tag{2.6}$$

Readers can refer to [3,4,6]. Knowing from Eq. (2.5) we can directly use the definition $\tilde{\Upsilon}_{ij}^{kl}$. Hence,

$$\Gamma = \sum_k \sum_{l=k}^{r} h_k^+ h_l^+ \sum_i \sum_{j=i}^{r} h_i h_j \tilde{\Upsilon}_{ij}^{kl},$$

where $\tilde{\Upsilon}_{ii}^{kk} = \Upsilon_{ii}^{kk}$; $\tilde{\Upsilon}_{ii}^{kl} = \Upsilon_{ii}^{kl} + \Upsilon_{ii}^{lk}, k \neq l$; $\tilde{\Upsilon}_{ij}^{kk} = \Upsilon_{ij}^{kk} + \Upsilon_{ji}^{kk}, i \neq j$; $\tilde{\Upsilon}_{ij}^{kl} = \Upsilon_{ij}^{kl} + \Upsilon_{ji}^{kl} + \Upsilon_{ij}^{lk} + \Upsilon_{ji}^{lk}$, $k \neq l, i \neq j$. Ref. [3] has utilized $\tilde{\Upsilon}_{ij}^{kl}$.

3. Homogeneously polynomially parameter-dependent solutions

Let us generalize the stability condition (2.6) as, for $q \geq 2$ and $p \geq 2$,

$$\Xi = \sum_{\mathbf{k} \in \mathbb{I}_q^+} h_{\mathbf{k}}^+ \sum_{\mathbf{i} \in \mathbb{I}_p^+} h_{\mathbf{i}} \tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}} > 0. \tag{3.1}$$

The left-hand side of (3.1) will be denoted as *upper triangular* $(q \times p)$-dimensional fuzzy summation. This is different from Eq. (9) of [25], a *full* p-dimensional fuzzy summation. Note that, when $q \leq 1$ or $p \leq 1$, there is no advantage to introduce $(q \times p)$-dimensional fuzzy summation, i.e., when $q \leq 1$ ($p \leq 1$), one should just define p-dimensional (q-dimensional) fuzzy summation.

Remark 1. By the upper triangular fuzzy summation, the number of decision variables can be minimized. By the full fuzzy summation, some decision variables are redundant. The upper triangular $(q \times p)$-dimensional fuzzy summation is defined over the membership functions at both t and $t+1$, while the full p-dimensional fuzzy summation in [25] is only defined over the membership functions at time t. The former is considerably less conservative than the latter. By the latter, we can only obtain the ANS stability conditions for the membership function-dependent model, but the Lyapunov matrix should be time-invariant. By the former, we can obtain the ANS stability conditions for both the membership function-dependent model and membership function-dependent Lyapunov matrix.

Lemma 1. *Let Eq. (2.1) be a discrete-time fuzzy model and consider HPP-non-PDC law*

$$u = -Y_{p-1}(h) S_{p-1}(h)^{-1} x. \tag{3.2}$$

The closed-loop fuzzy system is globally asymptotically stable, in the sense of HPNQ Lyapunov function

$$V(x) = x^{\mathrm{T}} S_{p-1}(h)^{-1} x, \tag{3.3}$$

if and only if there exist matrices $\tilde{Y}_{\mathbf{j}}$ and symmetric matrices $\tilde{S}_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{I}_{p-1}^+$ such that Eq. (3.1) holds with $q = p-1$, where

$$\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}} = \sum_{(i,\mathbf{j}) \in \mathcal{P}(\mathbf{i}),(s,\mathbf{v}) \in \mathcal{P}(i \in \mathbb{I}_1, \mathbf{j} \in \mathbb{I}_{p-1}^+)} \tilde{\upsilon}_{(s,\mathbf{v})}^{\mathbf{k}},$$

$$\tilde{\upsilon}^{\mathbf{k}}_{(i,\mathbf{j})} = \begin{bmatrix} \mathcal{C}_{p-1}(\mathbf{k})\tilde{S}_{\mathbf{j}} & (*) \\ \mathcal{C}_{p-1}(\mathbf{k})(A_i\tilde{S}_{\mathbf{j}} - B_i\tilde{Y}_{\mathbf{j}}) & \mathcal{C}_{p-1}(\mathbf{j})\tilde{S}_{\mathbf{k}} \end{bmatrix}, \quad \mathbf{j},\mathbf{k} \in \mathbb{I}^{+}_{p-1}, \quad i \in \mathbb{I}_1.$$

Proof. Refer to "Lemma 1" of [7]. □

If, in Lemma 1, one replaces $\{\tilde{Y}_{\mathbf{j}}, \tilde{S}_{\mathbf{j}}, \tilde{S}_{\mathbf{k}}\}$ with $\{Y_{\mathbf{j}}, S_{\mathbf{j}}, S_{\mathbf{k}}\}$ (removing "tilde" as in Eq. (2.4)), then much more decision matrices would be involved.

Lemma 2. *Let Eq. (2.1) be a discrete-time fuzzy model and consider HPP-non-PDC law*

$$u = -Y_{p-1}(h)G_{p-1}(h)^{-1}x. \tag{3.4}$$

The closed-loop fuzzy system is globally asymptotically stable, in the sense of HPNQ Lyapunov function

$$V(x) = x^{\mathrm{T}} S_p(h)^{-1} x, \tag{3.5}$$

if there exist matrices $\tilde{Y}_{\mathbf{j}}$, $\tilde{G}_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{I}^{+}_{p-1}$ and symmetric matrices $\tilde{S}_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}^{+}_{p}$ such that Eq. (3.1) holds with $q=p$, where

$$\tilde{\Upsilon}^{\mathbf{k}}_{\mathbf{i}} = \sum_{(i,\mathbf{j}) \in \mathcal{P}(\mathbf{i}),(s,\mathbf{v}) \in \mathcal{P}(i \in \mathbb{I}_1, \mathbf{j} \in \mathbb{I}^{+}_{p-1})} \tilde{\upsilon}^{\mathbf{k}}_{(s,\mathbf{v})},$$

$$\tilde{\upsilon}^{\mathbf{k}}_{(i,\mathbf{j})} = \begin{bmatrix} \mathcal{C}_p(\mathbf{k})\mathcal{C}_p(\mathbf{i})(\tilde{G}_{\mathbf{j}} + \tilde{G}^{\mathrm{T}}_{\mathbf{j}}) - \mathcal{C}_p(\mathbf{k})\mathcal{C}_{p-1}(\mathbf{j})\tilde{S}_{\mathbf{i}} & (*) \\ \mathcal{C}_p(\mathbf{k})\mathcal{C}_p(\mathbf{i})(A_i\tilde{G}_{\mathbf{j}} - B_i\tilde{Y}_{\mathbf{j}}) & \mathcal{C}_p(\mathbf{i})\mathcal{C}_{p-1}(\mathbf{j})\tilde{S}_{\mathbf{k}} \end{bmatrix},$$

$$\mathbf{i} \in \mathcal{P}((i,\mathbf{j})), \quad \mathbf{j} \in \mathbb{I}^{+}_{p-1}, \quad \mathbf{i},\mathbf{k} \in \mathbb{I}^{+}_{p}, \quad i \in \mathbb{I}_1. \tag{3.6}$$

Proof. Refer to "Lemma 2" of [7]. □

Example 3. Consider Eq. (3.6). (a) For $i=1$, $\mathbf{j}=(1,2)$, $\mathcal{P}(i \in \mathbb{I}_1, \mathbf{j} \in \mathbb{I}^{+}_{2}) = \{(1,1,2),(2,1,1)\}$. Further, let $\mathbf{k}=(1,1,1)$. Then

$$\tilde{\Upsilon}^{111}_{112} = \tilde{\upsilon}^{111}_{112} + \tilde{\upsilon}^{111}_{211} = \begin{bmatrix} 3\tilde{G}_{12} + 3\tilde{G}^{\mathrm{T}}_{12} - 2\tilde{S}_{112} & (*) \\ 3A_1\tilde{G}_{12} - 3B_1\tilde{Y}_{12} & 6\tilde{S}_{111} \end{bmatrix} + \begin{bmatrix} 3\tilde{G}_{11} + 3\tilde{G}^{\mathrm{T}}_{11} - \tilde{S}_{112} & (*) \\ 3A_2\tilde{G}_{11} - 3B_2\tilde{Y}_{11} & 3\tilde{S}_{111} \end{bmatrix}.$$

(b) For $i=1$, $\mathbf{j}=(2,3)$, $\mathcal{P}(i \in \mathbb{I}_1, \mathbf{j} \in \mathbb{I}^{+}_{2}) = \{(1,2,3),(2,1,3),(3,1,2)\}$. Further, let $\mathbf{k}=(1,2,2)$. Then

$$\tilde{\Upsilon}^{122}_{123} = \tilde{\upsilon}^{122}_{123} + \tilde{\upsilon}^{122}_{213} + \tilde{\upsilon}^{122}_{312} = \begin{bmatrix} 18\tilde{G}_{23} + 18\tilde{G}^{\mathrm{T}}_{23} - 6\tilde{S}_{123} & (*) \\ 18A_1\tilde{G}_{23} - 18B_1\tilde{Y}_{23} & 12\tilde{S}_{122} \end{bmatrix}$$
$$+ \begin{bmatrix} 18\tilde{G}_{13} + 18\tilde{G}^{\mathrm{T}}_{13} - 6\tilde{S}_{123} & (*) \\ 18A_2\tilde{G}_{13} - 18B_2\tilde{Y}_{13} & 12\tilde{S}_{122} \end{bmatrix} + \begin{bmatrix} 18\tilde{G}_{12} + 18\tilde{G}^{\mathrm{T}}_{12} - 6\tilde{S}_{123} & (*) \\ 18A_3\tilde{G}_{12} - 18B_3\tilde{Y}_{12} & 12\tilde{S}_{122} \end{bmatrix}.$$

It is noted that for the same p, Lemma 2 involves more decision variables, and is less conservative than Lemma 1 and that there exist $\check{S}_{i,\mathbf{j}}$ such that

$$\tilde{S}_{\mathbf{i}} = \sum_{(i,\mathbf{j}) \in \mathcal{P}(\mathbf{i}),(s,\mathbf{v}) \in \mathcal{P}(i \in \mathbb{I}_1, \mathbf{j} \in \mathbb{I}^{+}_{p-1})} \check{S}_{s,\mathbf{v}}.$$

By choosing $\breve{S}_{i,\mathbf{j}}=\tilde{S}_{\mathbf{j}}$ and $\tilde{G}_{\mathbf{j}}=\tilde{S}_{\mathbf{j}}$,

$$\sum_{\mathbf{i}\in\mathbb{I}_p^+} h_{\mathbf{i}}\tilde{S}_{\mathbf{i}}=\sum_{i\in\mathbb{I}_1} h_i \sum_{\mathbf{j}\in\mathbb{I}_{p-1}^+} h_{\mathbf{j}}\breve{S}_{i,\mathbf{j}}=\sum_{\mathbf{j}\in\mathbb{I}_{p-1}^+} h_{\mathbf{j}}\tilde{S}_{\mathbf{j}},$$

Lemma 2 reduces to Lemma 1. This point has been shown for the special case with $p=2$ in [14].

Lemma 3. *Let Eq. (2.1) be a discrete-time fuzzy model and consider HPPPDC law*

$$u=-F_{p-1}(h)x. \tag{3.7}$$

The closed-loop fuzzy system is globally asymptotically stable, in the sense of HPPD Lyapunov function

$$V(x)=x^{\mathrm{T}}P_p(h)x, \tag{3.8}$$

if and only if there exist symmetric matrices $\tilde{P}_{\mathbf{i}}$, $\tilde{P}_{\mathbf{k}}$ such that Eq. (3.1) holds with $q=p$, where

$$\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}}=\sum_{(i,\mathbf{j})\in\mathcal{P}(\mathbf{i}),(s,\mathbf{v})\in\mathcal{P}(i\in\mathbb{I}_1,\mathbf{j}\in\mathbb{I}_{p-1}^+)} \tilde{\upsilon}_{(s,\mathbf{v})}^{\mathbf{k}},$$

$$\tilde{\upsilon}_{(i,\mathbf{j})}^{\mathbf{k}}=\begin{bmatrix} \mathcal{C}_p(\mathbf{k})\mathcal{C}_{p-1}(\mathbf{j})\tilde{P}_{\mathbf{i}} & (*) \\ \mathcal{C}_p(\mathbf{i})\tilde{P}_{\mathbf{k}}(A_i-B_i\tilde{F}_{\mathbf{j}}) & \mathcal{C}_p(\mathbf{i})\mathcal{C}_{p-1}(\mathbf{j})\tilde{P}_{\mathbf{k}} \end{bmatrix},$$

$$\mathbf{i}\in\mathcal{P}((i,\mathbf{j})),\ \ \mathbf{j}\in\mathbb{I}_{p-1}^+,\ \ \mathbf{i},\mathbf{k}\in\mathbb{I}_p^+,\ \ i\in\mathbb{I}_1. \tag{3.9}$$

Proof. Refer to "Lemma 3" of [7]. □

For the special case with $q=p=2$, the Lyapunov functions of Lemmas 2 and 3 are referred to [3] and [6], respectively. For Lemmas 1–3, if Eq. (3.1) holds for some p_0, then it does for any $p\geq p_0$, i.e., increasing p reduces conservatism. Condition (3.1) is guaranteed by

$$\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}}>0,\quad \mathbf{i}\in\mathbb{I}_p^+,\ \ \mathbf{k}\in\mathbb{I}_q^+.$$

Polya's theorem and slack matrices can be utilized to further relax the above conditions, which will be done in the followed section.

Remark 2. For each fixed p, Lemmas 1–3 give the sufficient stability conditions. With the increase of p, any Lyapunov function continuous on the membership functions can be approximated by $x^{\mathrm{T}}P_p(h)x$ (see [8]). For Lemmas 1 and 2, with the increase of p, any control law continuous on the membership functions can be approximated by $u=-Y_{p-1}(h)S_{p-1}(h)^{-1}x$ or $u=-Y_{p-1}(h)G_{p-1}(h)^{-1}x$ (see [8]). Hence, theoretically, Lemmas 1 and 2 give the ANS stabilization conditions for any Lyapunov function and control law continuous on the membership functions, while Lemma 3 gives the ANS stability conditions for any Lyapunov function continuous on the membership functions. Although these viewpoints hold theoretically, they may be numerically intractable. In the following section, we introduce the slack matrices to improve the numerical efficiency.

4. Asymptotically necessary and sufficient stability conditions in the case of HPNQ Lyapunov functions

4.1. Relaxation by increasing the number of positiveness conditions

This subsection generalizes the results in [20,25], implicitly applying Polya's theorem. Consider Eq. (3.1). Proving $\Xi>0$ is equivalent to proving, for $m\geq q,\ n\geq p$

$$[\Xi]_n^m=\left(\sum_k h_k^+\right)^{m-q}\left(\sum_i h_i\right)^{n-p}\cdot\Xi>0.$$

Redefine $\mathbf{i}_p=(i_1,i_2,\ldots,i_p)$ and denote $\mathbf{i}_{p,n}=(i_{p+1},i_{p+2},\ldots,i_n)$. Then

$$[\Xi]_n^m=\sum_{\mathbf{k}_{q,m}\in\mathbb{I}_{m-q}}h^+_{\mathbf{k}_{q,m}}\sum_{\mathbf{i}_{p,n}\in\mathbb{I}_{n-p}}h_{\mathbf{i}_{p,n}}\left(\sum_{\mathbf{k}_q\in\mathbb{I}_q^+}h^+_{\mathbf{k}_q}\sum_{\mathbf{i}_p\in\mathbb{I}_p^+}h_{\mathbf{i}_p}\tilde{\Upsilon}^{\mathbf{k}_q}_{\mathbf{i}_q}\right)=\sum_{\mathbf{k}\in\mathbb{I}_m^+}h^+_{\mathbf{k}}\sum_{\mathbf{i}\in\mathbb{I}_n^+}h_{\mathbf{i}}\tilde{\Upsilon}^{\mathbf{k}}_{\mathbf{i}},\tag{4.1}$$

where

$$\tilde{\Upsilon}^{\mathbf{k}}_{\mathbf{i}}=\sum_{(\mathbf{l}_q\mathbf{l}_{q,m})\in\mathcal{P}(\mathbf{k}_q\in\mathbb{I}_q^+,\mathbf{k}_{q,m}\in\mathbb{I}_{m-q})}\ \sum_{(\mathbf{j}_p\mathbf{j}_{p,n})\in\mathcal{P}(\mathbf{i}_p\in\mathbb{I}_p^+,\mathbf{i}_{p,n}\in\mathbb{I}_{n-p})}\tilde{\Upsilon}^{\mathbf{l}_q}_{\mathbf{j}_p}.$$

Note that, in Eq. (4.1), $\mathbf{j}_{p,n}=(j_{p+1},j_{p+2},\ldots,j_n)$, $\mathbf{k}_{q,m}=(k_{q+1},k_{q+2},\ldots,k_m)$, $\mathbf{l}_{q,m}=(l_{q+1},l_{q+2},\ldots,l_m)$, $\mathbf{j}_p=(j_1,j_2,\ldots,j_p)$, $\mathbf{k}_q=(k_1,k_2,\ldots,k_q)$, $\mathbf{l}_q=(l_1,l_2,\ldots,l_q)$.

Theorem 1. *For the fixed* $\{m,n\}$, *the positive-definite conditions*

$$\tilde{\Upsilon}^{\mathbf{k}}_{\mathbf{i}}>0,\quad \mathbf{i}\in\mathbb{I}_n^+,\ \ \mathbf{k}\in\mathbb{I}_m^+\tag{4.2}$$

are sufficient for positiveness of Ξ *in Eq. (3.1). Moreover, if Eq. (3.1) holds, then there exists finite* $\{m,n\}$ *such that Eq. (4.2) holds; if Eq. (4.2) holds for some* $\{m_0,n_0\}$, *then it does for any* $\{m\geq m_0,n\geq n_0\}$.

Proof. This conclusion can be proved by combining the procedures in "Proposition 1", "Proposition 2" and "Theorem 1" of [25]. Refer to "Theorem 1" of [7]. □

Note that, the above result is different in two aspects from [25]: (i) dimensionality ($q\times p$); (ii) calculation based on the *upper triangular* fuzzy summation in Eq. (3.1). By (i), this paper can obtain the ANS stability conditions for the time-varying Lyapunov matrix. By (ii), the number of decision variables is minimized for each specified set of stability conditions.

Example 4. Consider Eq. (4.2). Let $\mathbf{i}=(1,1,2,2),\mathbf{k}=(1,1,2,3)$. Then

$$\mathcal{P}(\mathbf{i}_2\in\mathbb{I}_2^+,\mathbf{i}_{2,2}\in\mathbb{I}_2)=\{(1,1,2,2),(1,2,1,2),(1,2,2,1),(2,2,1,1)\},$$
$$\mathcal{P}(\mathbf{k}_2\in\mathbb{I}_2^+,\mathbf{k}_{2,2}\in\mathbb{I}_2)=\{(1,1,2,3),(1,1,3,2),(1,2,1,3),(1,2,3,1),$$
$$(1,3,1,2),(1,3,2,1),(2,3,1,1)\},$$
$$\tilde{\Upsilon}^{1123}_{1122}=\sum_{(l_1,l_2,l_3,l_4)\in\mathcal{P}((k_1,k_2)\in\mathbb{I}_2^+,(k_3,k_4)\in\mathbb{I}_2)}\ \sum_{(j_1,j_2,j_3,j_4)\in\mathcal{P}((i_1,i_2)\in\mathbb{I}_2^+,(i_3,i_4)\in\mathbb{I}_2)}\tilde{\Upsilon}^{l_1l_2}_{j_1j_2}$$
$$=\sum_{(l_1,l_2,l_3,l_4)\in\mathcal{P}((k_1,k_2)\in\mathbb{I}_2^+,(k_3,k_4)\in\mathbb{I}_2)}(\tilde{\Upsilon}^{l_1l_2}_{11}+2\tilde{\Upsilon}^{l_1l_2}_{12}+\tilde{\Upsilon}^{l_1l_2}_{22})$$
$$=2(\tilde{\Upsilon}^{11}_{11}+2\tilde{\Upsilon}^{11}_{12}+\tilde{\Upsilon}^{11}_{22})+2(\tilde{\Upsilon}^{12}_{11}+2\tilde{\Upsilon}^{12}_{12}+\tilde{\Upsilon}^{12}_{22})+2(\tilde{\Upsilon}^{13}_{11}+2\tilde{\Upsilon}^{13}_{12}+\tilde{\Upsilon}^{13}_{22})+(\tilde{\Upsilon}^{23}_{11}+2\tilde{\Upsilon}^{23}_{12}+\tilde{\Upsilon}^{23}_{22}).$$

4.2. Relaxation via slack variables

This section generalizes the results in [25].

Theorem 2. *Consider proving* $\Xi > 0$ *via* $[\Xi]_n^m > 0$, $m > 2, n > 2$. *Then*

- *a sufficient condition for* $\Xi > 0$ *is*

$$\sum_{\mathbf{k} \in \mathbb{I}_m^+} h_{\mathbf{k}}^+ \sum_{\mathbf{i} \in \mathbb{I}_{n-2}^+} h_{\mathbf{i}} \begin{bmatrix} \tilde{X}_{(\mathbf{i},1,1)}^{\mathbf{k}} & \tilde{X}_{(\mathbf{i},1,2)}^{\mathbf{k}} & \cdots & \tilde{X}_{(\mathbf{i},1,r)}^{\mathbf{k}} \\ \tilde{X}_{(\mathbf{i},2,1)}^{\mathbf{k}} & \tilde{X}_{(\mathbf{i},2,2)}^{\mathbf{k}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{X}_{(\mathbf{i},r-1,r)}^{\mathbf{k}} \\ \tilde{X}_{(\mathbf{i},r,1)}^{\mathbf{k}} & \cdots & \tilde{X}_{(\mathbf{i},r,r-1)}^{\mathbf{k}} & \tilde{X}_{(\mathbf{i},r,r)}^{\mathbf{k}} \end{bmatrix} > 0 \tag{4.3}$$

for appropriately defined matrices $\tilde{X}_{(\mathbf{i},a,b)}^{\mathbf{k}} = (\tilde{X}_{(\mathbf{i},b,a)}^{\mathbf{k}})^{\mathrm{T}} \in \Re^{2n_x \times 2n_x}$, *if*

$$\tilde{Y}_{\mathbf{i}}^{\mathbf{k}} > \sum_{(\mathbf{j}_{n-2}, \mathbf{j}_{n-2,n}) \in \mathcal{P}(\mathbf{i}_{n-2} \in \mathbb{I}_{n-2}^+, \mathbf{i}_{n-2,n} \in \mathbb{I}_2)} \tilde{X}_{\mathbf{j}}^{\mathbf{k}}, \quad \forall \mathbf{i} \in \mathbb{I}_n^+, \mathbf{k} \in \mathbb{I}_m^+. \tag{4.4}$$

Moreover, if Eq. (3.1) *holds, then there exist finite* $\{m, n\}$ *such that Eqs.* (4.3) *and* (4.4) *hold; if Eqs.* (4.3) *and* (4.4) *hold for some* $\{m_0, n_0\}$, *then they do for any* $\{m \geq m_0, n \geq n_0\}$;

- *a sufficient condition for Eq.* (4.3) *is*

$$\sum_{\mathbf{k} \in \mathbb{I}_{m-2}^+} h_{\mathbf{k}}^+ \sum_{\mathbf{i} \in \mathbb{I}_{n-2}^+} h_{\mathbf{i}} \begin{bmatrix} \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},1,1)} & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},1,2)} & \cdots & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},1,r)} \\ \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},2,1)} & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},2,2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r-1,r)} \\ \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r,1)} & \cdots & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r,r-1)} & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r,r)} \end{bmatrix} > 0 \tag{4.5}$$

for appropriately defined matrices $\tilde{Y}_{\mathbf{i}}^{(\mathbf{k},a,b)} = (\tilde{Y}_{\mathbf{i}}^{(\mathbf{k},b,a)})^{\mathrm{T}} \in \Re^{2rn_x \times 2rn_x}$, *if*

$$\begin{bmatrix} \tilde{X}_{(\mathbf{i},1,1)}^{\mathbf{k}} & \tilde{X}_{(\mathbf{i},1,2)}^{\mathbf{k}} & \cdots & \tilde{X}_{(\mathbf{i},1,r)}^{\mathbf{k}} \\ \tilde{X}_{(\mathbf{i},2,1)}^{\mathbf{k}} & \tilde{X}_{(\mathbf{i},2,2)}^{\mathbf{k}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{X}_{(\mathbf{i},r-1,r)}^{\mathbf{k}} \\ \tilde{X}_{(\mathbf{i},r,1)}^{\mathbf{k}} & \cdots & \tilde{X}_{(\mathbf{i},r,r-1)}^{\mathbf{k}} & \tilde{X}_{(\mathbf{i},r,r)}^{\mathbf{k}} \end{bmatrix} > \sum_{(\mathbf{l}_{m-2}, \mathbf{l}_{m-2,m}) \in \mathcal{P}(\mathbf{k}_{m-2} \in \mathbb{I}_{m-2}^+, \mathbf{k}_{m-2,m} \in \mathbb{I}_2)} \tilde{Y}_{\mathbf{i}}^{\mathbf{l}},$$
$$\forall \mathbf{i} \in \mathbb{I}_{n-2}^+, \quad \mathbf{k} \in \mathbb{I}_m^+. \tag{4.6}$$

Moreover, if Eq. (3.1) *holds, then there exist finite* $\{m, n\}$ *such that Eqs.* (4.4)–(4.6) *hold; if Eqs.* (4.4)–(4.6) *hold for some* $\{m_0, n_0\}$, *then they do for any* $\{m \geq m_0, n \geq n_0\}$.

Theorem 2 is based on Theorem 1. Compared with Theorem 1, the slack matrices are introduced in Theorem 2 in order to improve the numerical efficiency. The improvement of the numerical efficiency by applying the slack matrices has been exposed in the early works [9,16,19], which is not repeated here.

Note that, there is a major difference between Theorem 2 here and “Theorem 4” of [25], i.e., both Eqs. (4.3) and (4.5) are upper triangular fuzzy summations while [25] uses the full fuzzy summation. Use of upper triangular summation can greatly reduce the number of inequalities.

By applying Theorem 2, a set of sufficient LMI conditions for $\Xi > 0$ is any one of the following:

$$\begin{bmatrix} \tilde{X}^{\mathbf{k}}_{(\mathbf{i},1,1)} & \tilde{X}^{\mathbf{k}}_{(\mathbf{i},1,2)} & \cdots & \tilde{X}^{\mathbf{k}}_{(\mathbf{i},1,r)} \\ \tilde{X}^{\mathbf{k}}_{(\mathbf{i},2,1)} & \tilde{X}^{\mathbf{k}}_{(\mathbf{i},2,2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{X}^{\mathbf{k}}_{(\mathbf{i},r-1,r)} \\ \tilde{X}^{\mathbf{k}}_{(\mathbf{i},r,1)} & \cdots & \tilde{X}^{\mathbf{k}}_{(\mathbf{i},r,r-1)} & \tilde{X}^{\mathbf{k}}_{(\mathbf{i},r,r)} \end{bmatrix} > 0, \quad \mathbf{i} \in \mathbb{I}^+_{n-2}, \ \ \mathbf{k} \in \mathbb{I}^+_{m}, \ \ \text{Eq. (4.4)}, \tag{4.7}$$

$$\begin{bmatrix} \tilde{Y}^{(\mathbf{k},1,1)}_{\mathbf{i}} & \tilde{Y}^{(\mathbf{k},1,2)}_{\mathbf{i}} & \cdots & \tilde{Y}^{(\mathbf{k},1,r)}_{\mathbf{i}} \\ \tilde{Y}^{(\mathbf{k},2,1)}_{\mathbf{i}} & \tilde{Y}^{(\mathbf{k},2,2)}_{\mathbf{i}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{Y}^{(\mathbf{k},r-1,r)}_{\mathbf{i}} \\ \tilde{Y}^{(\mathbf{k},r,1)}_{\mathbf{i}} & \cdots & \tilde{Y}^{(\mathbf{k},r,r-1)}_{\mathbf{i}} & \tilde{Y}^{(\mathbf{k},r,r)}_{\mathbf{i}} \end{bmatrix} > 0, \quad \mathbf{i} \in \mathbb{I}^+_{n-2}, \ \ \mathbf{k} \in \mathbb{I}^+_{m-2}, \ \ \text{Eq. (4.4)}, \ \ \text{Eq. (4.6)}. \tag{4.8}$$

Example 5. Consider $m = n = 4$, $n \geq \max\{q, p\}$, $\min\{q, p\} \geq 2$ and $r = 2$. Then Eq. (4.8) is expanded as below

(a)

$$\tilde{Y}^{1111}_{1111} > \tilde{X}^{1111}_{1111}, \quad \tilde{Y}^{1112}_{1111} > \tilde{X}^{1112}_{1111}, \quad \tilde{Y}^{1122}_{1111} > \tilde{X}^{1122}_{1111}, \quad \tilde{Y}^{1222}_{1111} > \tilde{X}^{1222}_{1111}, \quad \tilde{Y}^{2222}_{1111} > \tilde{X}^{2222}_{1111},$$
$$\tilde{Y}^{1111}_{1112} > \tilde{X}^{1111}_{1112} + \tilde{X}^{1111}_{1121} + \tilde{X}^{1111}_{1211}, \quad \tilde{Y}^{1112}_{1112} > \tilde{X}^{1112}_{1112} + \tilde{X}^{1112}_{1121} + \tilde{X}^{1112}_{1211},$$
$$\tilde{Y}^{1122}_{1112} > \tilde{X}^{1122}_{1112} + \tilde{X}^{1122}_{1121} + \tilde{X}^{1122}_{1211},$$
$$\tilde{Y}^{1222}_{1112} > \tilde{X}^{1222}_{1112} + \tilde{X}^{1222}_{1121} + \tilde{X}^{1222}_{1211}, \quad \tilde{Y}^{2222}_{1112} > \tilde{X}^{2222}_{1112} + \tilde{X}^{2222}_{1121} + \tilde{X}^{2222}_{1211},$$
$$\tilde{Y}^{1111}_{1122} > \tilde{X}^{1111}_{1122} + \tilde{X}^{1111}_{1212} + \tilde{X}^{1111}_{1221} + \tilde{X}^{1111}_{2211}, \quad \tilde{Y}^{1112}_{1122} > \tilde{X}^{1112}_{1122} + \tilde{X}^{1112}_{1212} + \tilde{X}^{1112}_{1221} + \tilde{X}^{1112}_{2211},$$
$$\tilde{Y}^{1122}_{1122} > \tilde{X}^{1122}_{1122} + \tilde{X}^{1122}_{1212} + \tilde{X}^{1122}_{1221} + \tilde{X}^{1122}_{2211}, \quad \tilde{Y}^{1222}_{1122} > \tilde{X}^{1222}_{1122} + \tilde{X}^{1222}_{1212} + \tilde{X}^{1222}_{1221} + \tilde{X}^{1222}_{2211},$$
$$\tilde{Y}^{2222}_{1122} > \tilde{X}^{2222}_{1122} + \tilde{X}^{2222}_{1212} + \tilde{X}^{2222}_{1221} + \tilde{X}^{2222}_{2211},$$
$$\tilde{Y}^{1111}_{1222} > \tilde{X}^{1111}_{1222} + \tilde{X}^{1111}_{2212} + \tilde{X}^{1111}_{2221}, \quad \tilde{Y}^{1112}_{1222} > \tilde{X}^{1112}_{1222} + \tilde{X}^{1112}_{2212} + \tilde{X}^{1112}_{2221},$$
$$\tilde{Y}^{1122}_{1222} > \tilde{X}^{1122}_{1222} + \tilde{X}^{1122}_{2212} + \tilde{X}^{1122}_{2221},$$
$$\tilde{Y}^{1222}_{1222} > \tilde{X}^{1222}_{1222} + \tilde{X}^{1222}_{2212} + \tilde{X}^{1222}_{2221}, \quad \tilde{Y}^{2222}_{1222} > \tilde{X}^{2222}_{1222} + \tilde{X}^{2222}_{2212} + \tilde{X}^{2222}_{2221},$$
$$\tilde{Y}^{1111}_{2222} > \tilde{X}^{1111}_{2222}, \quad \tilde{Y}^{1112}_{2222} > \tilde{X}^{1112}_{2222}, \quad \tilde{Y}^{1122}_{2222} > \tilde{X}^{1122}_{2222}, \quad \tilde{Y}^{1222}_{2222} > \tilde{X}^{1222}_{2222}, \quad \tilde{Y}^{2222}_{2222} > \tilde{X}^{2222}_{2222};$$

(b)

$$\begin{bmatrix} \tilde{X}^{1111}_{1111} & \tilde{X}^{1111}_{1112} \\ \tilde{X}^{1111}_{1121} & \tilde{X}^{1111}_{1122} \end{bmatrix} > \tilde{Y}^{1111}_{11}, \quad \begin{bmatrix} \tilde{X}^{1112}_{1111} & \tilde{X}^{1112}_{1112} \\ \tilde{X}^{1112}_{1121} & \tilde{X}^{1112}_{1122} \end{bmatrix} > \tilde{Y}^{1112}_{11} + \tilde{Y}^{1121}_{11} + \tilde{Y}^{1211}_{11},$$

$$\begin{bmatrix} \tilde{X}^{1122}_{1111} & \tilde{X}^{1122}_{1112} \\ \tilde{X}^{1122}_{1121} & \tilde{X}^{1122}_{1122} \end{bmatrix} > \tilde{Y}^{1122}_{11} + \tilde{Y}^{1212}_{11} + \tilde{Y}^{1221}_{11} + \tilde{Y}^{2211}_{11},$$

$$\begin{bmatrix} \tilde{X}_{1111}^{1222} & \tilde{X}_{1112}^{1222} \\ \tilde{X}_{1121}^{1222} & \tilde{X}_{1122}^{1222} \end{bmatrix} > \tilde{Y}_{11}^{1222} + \tilde{Y}_{11}^{2212} + \tilde{Y}_{11}^{2221}, \quad \begin{bmatrix} \tilde{X}_{1111}^{2222} & \tilde{X}_{1112}^{2222} \\ \tilde{X}_{1121}^{2222} & \tilde{X}_{1122}^{2222} \end{bmatrix} > \tilde{Y}_{11}^{2222},$$

$$\begin{bmatrix} \tilde{X}_{1211}^{1111} & \tilde{X}_{1212}^{1111} \\ \tilde{X}_{1221}^{1111} & \tilde{X}_{1222}^{1111} \end{bmatrix} > \tilde{Y}_{12}^{1111}, \quad \begin{bmatrix} \tilde{X}_{1211}^{1112} & \tilde{X}_{1212}^{1112} \\ \tilde{X}_{1221}^{1112} & \tilde{X}_{1222}^{1112} \end{bmatrix} > \tilde{Y}_{12}^{1112} + \tilde{Y}_{12}^{1121} + \tilde{Y}_{12}^{1211},$$

$$\begin{bmatrix} \tilde{X}_{1211}^{1122} & \tilde{X}_{1212}^{1122} \\ \tilde{X}_{1221}^{1122} & \tilde{X}_{1222}^{1122} \end{bmatrix} > \tilde{Y}_{12}^{1122} + \tilde{Y}_{12}^{1212} + \tilde{Y}_{12}^{1221} + \tilde{Y}_{12}^{2211},$$

$$\begin{bmatrix} \tilde{X}_{1211}^{1222} & \tilde{X}_{1212}^{1222} \\ \tilde{X}_{1221}^{1222} & \tilde{X}_{1222}^{1222} \end{bmatrix} > \tilde{Y}_{12}^{1222} + \tilde{Y}_{12}^{2212} + \tilde{Y}_{12}^{2221}, \quad \begin{bmatrix} \tilde{X}_{1211}^{2222} & \tilde{X}_{1212}^{2222} \\ \tilde{X}_{1221}^{2222} & \tilde{X}_{1222}^{2222} \end{bmatrix} > \tilde{Y}_{12}^{2222},$$

$$\begin{bmatrix} \tilde{X}_{2211}^{1111} & \tilde{X}_{2212}^{1111} \\ \tilde{X}_{2221}^{1111} & \tilde{X}_{2222}^{1111} \end{bmatrix} > \tilde{Y}_{22}^{1111}, \quad \begin{bmatrix} \tilde{X}_{2211}^{1112} & \tilde{X}_{2212}^{1112} \\ \tilde{X}_{2221}^{1112} & \tilde{X}_{2222}^{1112} \end{bmatrix} > \tilde{Y}_{22}^{1112} + \tilde{Y}_{22}^{1121} + \tilde{Y}_{22}^{1211},$$

$$\begin{bmatrix} \tilde{X}_{2211}^{1122} & \tilde{X}_{2212}^{1122} \\ \tilde{X}_{2221}^{1122} & \tilde{X}_{2222}^{1122} \end{bmatrix} > \tilde{Y}_{22}^{1122} + \tilde{Y}_{22}^{1212} + \tilde{Y}_{22}^{1221} + \tilde{Y}_{22}^{2211},$$

$$\begin{bmatrix} \tilde{X}_{2211}^{1222} & \tilde{X}_{2212}^{1222} \\ \tilde{X}_{2221}^{1222} & \tilde{X}_{2222}^{1222} \end{bmatrix} > \tilde{Y}_{22}^{1222} + \tilde{Y}_{22}^{2212} + \tilde{Y}_{22}^{2221}, \quad \begin{bmatrix} \tilde{X}_{2211}^{2222} & \tilde{X}_{2212}^{2222} \\ \tilde{X}_{2221}^{2222} & \tilde{X}_{2222}^{2222} \end{bmatrix} > \tilde{Y}_{22}^{2222};$$

(c)

$$\begin{bmatrix} \tilde{Y}_{11}^{1111} & \tilde{Y}_{11}^{1112} \\ \tilde{Y}_{11}^{1121} & \tilde{Y}_{11}^{1122} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{Y}_{11}^{1211} & \tilde{Y}_{11}^{1212} \\ \tilde{Y}_{11}^{1221} & \tilde{Y}_{11}^{1222} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{Y}_{11}^{2211} & \tilde{Y}_{11}^{2212} \\ \tilde{Y}_{11}^{2221} & \tilde{Y}_{11}^{2222} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \tilde{Y}_{12}^{1111} & \tilde{Y}_{12}^{1112} \\ \tilde{Y}_{12}^{1121} & \tilde{Y}_{12}^{1122} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{Y}_{12}^{1211} & \tilde{Y}_{12}^{1212} \\ \tilde{Y}_{12}^{1221} & \tilde{Y}_{12}^{1222} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{Y}_{12}^{2211} & \tilde{Y}_{12}^{2212} \\ \tilde{Y}_{12}^{2221} & \tilde{Y}_{12}^{2222} \end{bmatrix} > 0,$$

$$\begin{bmatrix} \tilde{Y}_{22}^{1111} & \tilde{Y}_{22}^{1112} \\ \tilde{Y}_{22}^{1121} & \tilde{Y}_{22}^{1122} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{Y}_{22}^{1211} & \tilde{Y}_{22}^{1212} \\ \tilde{Y}_{22}^{1221} & \tilde{Y}_{22}^{1222} \end{bmatrix} > 0, \quad \begin{bmatrix} \tilde{Y}_{22}^{2211} & \tilde{Y}_{22}^{2212} \\ \tilde{Y}_{22}^{2221} & \tilde{Y}_{22}^{2222} \end{bmatrix} > 0.$$

In the case of Lemma 2 with $q = p = 3$, the decision variables for (a)–(c) are

(A) any $Y_{11}, Y_{12}, Y_{22}, G_{11}, G_{12}, G_{22}$, symmetric $S_{111}, S_{112}, S_{122}, S_{222}$;

(B) symmetric $\tilde{X}_{1111}^{1111}, \tilde{X}_{1111}^{1112}, \tilde{X}_{1111}^{1122}, \tilde{X}_{1111}^{1222}, \tilde{X}_{1111}^{2222}$,
any
$\tilde{X}_{1112}^{1111} = (\tilde{X}_{1121}^{1111})^{\mathrm{T}},\ \tilde{X}_{1112}^{1112} = (\tilde{X}_{1121}^{1112})^{\mathrm{T}},\ \tilde{X}_{1112}^{1122} = (\tilde{X}_{1121}^{1122})^{\mathrm{T}},\ \tilde{X}_{1112}^{1222} = (\tilde{X}_{1121}^{1222})^{\mathrm{T}},\ \tilde{X}_{1112}^{2222} = (\tilde{X}_{1121}^{2222})^{\mathrm{T}}$,
symmetric $\tilde{X}_{1211}^{1111},\ \tilde{X}_{1211}^{1112},\ \tilde{X}_{1211}^{1122},\ \tilde{X}_{1211}^{1222},\ \tilde{X}_{1211}^{2222}$,
symmetric $\tilde{X}_{1122}^{1111},\ \tilde{X}_{1122}^{1112},\ \tilde{X}_{1122}^{1122},\ \tilde{X}_{1122}^{1222},\ \tilde{X}_{1122}^{2222}$,
any
$\tilde{X}_{1212}^{1111} = (\tilde{X}_{1221}^{1111})^{\mathrm{T}},\ \tilde{X}_{1212}^{1112} = (\tilde{X}_{1221}^{1112})^{\mathrm{T}},\ \tilde{X}_{1212}^{1122} = (\tilde{X}_{1221}^{1122})^{\mathrm{T}},\ \tilde{X}_{1212}^{1222} = (\tilde{X}_{1221}^{1222})^{\mathrm{T}},\ \tilde{X}_{1212}^{2222} = (\tilde{X}_{1221}^{2222})^{\mathrm{T}}$,

symmetric $\tilde{X}_{2211}^{1111},\ \tilde{X}_{2211}^{1112},\ \tilde{X}_{2211}^{1122},\ \tilde{X}_{2211}^{1222},\ \tilde{X}_{2211}^{2222}$,
symmetric $\tilde{X}_{1222}^{1111},\ \tilde{X}_{1222}^{1112},\ \tilde{X}_{1222}^{1122},\ \tilde{X}_{1222}^{1222},\ \tilde{X}_{1222}^{2222}$,
any
$\tilde{X}_{2212}^{1111}=(\tilde{X}_{2221}^{1111})^{\mathrm{T}},\ \tilde{X}_{2212}^{1112}=(\tilde{X}_{2221}^{1112})^{\mathrm{T}},\ \tilde{X}_{2212}^{1122}=(\tilde{X}_{2221}^{1122})^{\mathrm{T}},\ \tilde{X}_{2212}^{1222}=(\tilde{X}_{2221}^{1222})^{\mathrm{T}},\ \tilde{X}_{2212}^{2222}=(\tilde{X}_{2221}^{2222})^{\mathrm{T}}$,
symmetric $\tilde{X}_{2222}^{1111},\ \tilde{X}_{2222}^{1112},\ \tilde{X}_{2222}^{1122},\ \tilde{X}_{2222}^{1222},\ \tilde{X}_{2222}^{2222}$;

(C) symmetric $\tilde{Y}_{11}^{1111},\ \tilde{Y}_{11}^{1211},\ \tilde{Y}_{11}^{1122},\ \tilde{Y}_{11}^{2211},\ \tilde{Y}_{11}^{1222},\ \tilde{Y}_{11}^{2222}$,
any $\tilde{Y}_{11}^{1112}=(\tilde{Y}_{11}^{1121})^{\mathrm{T}},\ \tilde{Y}_{11}^{1212}=(\tilde{Y}_{11}^{1221})^{\mathrm{T}},\ \tilde{Y}_{11}^{2212}=(\tilde{Y}_{11}^{2221})^{\mathrm{T}}$,
symmetric $\tilde{Y}_{12}^{1111},\ \tilde{Y}_{12}^{1211},\ \tilde{Y}_{12}^{1122},\ \tilde{Y}_{12}^{2211},\ \tilde{Y}_{12}^{1222},\ \tilde{Y}_{12}^{2222}$,
any $\tilde{Y}_{12}^{1112}=(\tilde{Y}_{12}^{1121})^{\mathrm{T}},\ \tilde{Y}_{12}^{1212}=(\tilde{Y}_{12}^{1221})^{\mathrm{T}},\ \tilde{Y}_{12}^{2212}=(\tilde{Y}_{12}^{2221})^{\mathrm{T}}$,
symmetric $\tilde{Y}_{22}^{1111},\ \tilde{Y}_{22}^{1211},\ \tilde{Y}_{22}^{1122},\ \tilde{Y}_{22}^{2211},\ \tilde{Y}_{22}^{1222},\ \tilde{Y}_{22}^{2222}$,
any $\tilde{Y}_{22}^{1112}=(\tilde{Y}_{22}^{1121})^{\mathrm{T}},\ \tilde{Y}_{22}^{1212}=(\tilde{Y}_{22}^{1221})^{\mathrm{T}},\ \tilde{Y}_{22}^{2212}=(\tilde{Y}_{22}^{2221})^{\mathrm{T}}$.

Based on the above results, the following algorithm to compute sufficient conditions for stability of the closed loop system (i.e. proving $\Xi>0$) can be developed.

Algorithm 1. For any desired value of the complexity parameters $m=n$

(1) (Initialization) Obtain Eq. (4.1) with $m=n$, $n\geq\max\{q,p\}$. Set $\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}[0]}=\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}}$. The dimension of the multi-indices in iteration step τ will be denoted as $(2\times d_\tau)$, starting with $d_0=n$.
(2) (Recursive procedure) Consider now iteration step $\tau\geq 0$, trying to set up sufficient conditions for

$$\sum_{\mathbf{k}\in\mathbb{I}_{d_\tau}^{+}} h_{\mathbf{k}}^{+}\sum_{\mathbf{i}\in\mathbb{I}_{d_\tau}^{+}} h_{\mathbf{i}}\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}[\tau]}>0, \tag{4.9}$$

with $d_\tau>2$. Then, a sufficient condition for Eq. (4.9) is

$$\sum_{\mathbf{k}\in\mathbb{I}_{d_\tau-2}^{+}} h_{\mathbf{k}}^{+}\sum_{\mathbf{i}\in\mathbb{I}_{d_\tau-2}^{+}} h_{\mathbf{i}}\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}[\tau+1]}>0 \tag{4.10}$$

for appropriately defined matrices

$$\tilde{Y}_{\mathbf{i}}^{(\mathbf{k},a,b)[\tau]}=(\tilde{Y}_{\mathbf{i}}^{(\mathbf{k},b,a)[\tau]})^{\mathrm{T}}\in\Re^{(2r^{2\tau+1}n_x)\times(2r^{2\tau+1}n_x)},\qquad \tilde{X}_{(\mathbf{i},a,b)}^{\mathbf{k}[\tau]}=(\tilde{X}_{(\mathbf{i},b,a)}^{\mathbf{k}[\tau]})^{\mathrm{T}}\in\Re^{(2r^{2\tau}n_x)\times(2r^{2\tau}n_x)},$$

if

$$\tilde{\Upsilon}_{\mathbf{i}}^{\mathbf{k}[\tau]}>\sum_{(\mathbf{j}_{d_\tau-2},\mathbf{j}_{d_\tau-2,d_\tau})\in\mathcal{P}(\mathbf{i}_{d_\tau-2}\in\mathbb{I}_{d_\tau-2}^{+},\mathbf{i}_{d_\tau-2,d_\tau}\in\mathbb{I}_2)}\tilde{X}_{\mathbf{j}}^{\mathbf{k}[\tau]},\quad\forall\mathbf{i},\mathbf{k}\in\mathbb{I}_{d_\tau}^{+}, \tag{4.11}$$

$$\begin{bmatrix}\tilde{X}_{(\mathbf{i},1,1)}^{\mathbf{k}[\tau]} & \tilde{X}_{(\mathbf{i},1,2)}^{\mathbf{k}[\tau]} & \cdots & \tilde{X}_{(\mathbf{i},1,r)}^{\mathbf{k}[\tau]}\\ \tilde{X}_{(\mathbf{i},2,1)}^{\mathbf{k}[\tau]} & \tilde{X}_{(\mathbf{i},2,2)}^{\mathbf{k}[\tau]} & \ddots & \vdots\\ \vdots & \ddots & \ddots & \tilde{X}_{(\mathbf{i},r-1,r)}^{\mathbf{k}[\tau]}\\ \tilde{X}_{(\mathbf{i},r,1)}^{\mathbf{k}[\tau]} & \cdots & \tilde{X}_{(\mathbf{i},r,r-1)}^{\mathbf{k}[\tau]} & \tilde{X}_{(\mathbf{i},r,r)}^{\mathbf{k}[\tau]}\end{bmatrix}>\sum_{(\mathbf{l}_{d_\tau-2},\mathbf{l}_{d_\tau-2,d_\tau})\in\mathcal{P}(\mathbf{k}_{d_\tau-2}\in\mathbb{I}_{d_\tau-2}^{+},\mathbf{k}_{d_\tau-2,d_\tau}\in\mathbb{I}_2)}\tilde{Y}_{\mathbf{i}}^{\mathbf{l}[\tau]},$$

$$\forall\mathbf{i}\in\mathbb{I}_{d_\tau-2}^{+},\quad\mathbf{k}\in\mathbb{I}_{d_\tau}^{+}, \tag{4.12}$$

$$\tilde{Y}_{\mathbf{i}}^{\mathbf{k}[\tau+1]} = \begin{bmatrix} \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},1,1)[\tau]} & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},1,2)[\tau]} & \cdots & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},1,r)[\tau]} \\ \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},2,1)[\tau]} & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},2,2)[\tau]} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r-1,r)[\tau]} \\ \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r,1)[\tau]} & \cdots & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r,r-1)[\tau]} & \tilde{Y}_{\mathbf{i}}^{(\mathbf{k},r,r)[\tau]} \end{bmatrix}, \quad \forall \mathbf{i} \in \mathbb{I}_{d_\tau-2}^{+}, \quad \mathbf{k} \in \mathbb{I}_{d_\tau-2}^{+}. \tag{4.13}$$

Let $d_{\tau+1} = d_\tau - 2$. This procedure is proceeded recursively for $\tau = 0, 1, \ldots, \tau_{\max} - 1$ with $d_{\tau_{\max}} = 1$ or $d_{\tau_{\max}} = 2$ depending on whether n is odd or even, respectively.

(3) If $d_{\tau_{\max}} = 1$, then a sufficient condition for $\sum_k h_k^+ \sum_i h_i \tilde{Y}_i^{k[\tau_{\max}]} > 0$ is

$$\tilde{Y}_i^{k[\tau_{\max}]} > 0, \quad i, k = 1, \ldots, r. \tag{4.14}$$

If $d_{\tau_{\max}} = 2$, then a sufficient condition for $\sum_{\mathbf{k} \in \mathbb{I}_2^+} h_{\mathbf{k}}^+ \sum_{\mathbf{i} \in \mathbb{I}_2^+} h_{\mathbf{i}} \tilde{Y}_{\mathbf{i}}^{\mathbf{k}[\tau_{\max}]} > 0$ is

$$\begin{bmatrix} \tilde{Y}^{11[\tau_{\max}]} & \tilde{Y}^{12[\tau_{\max}]} & \cdots & \tilde{Y}^{1r[\tau_{\max}]} \\ \tilde{Y}^{21[\tau_{\max}]} & \tilde{Y}^{22[\tau_{\max}]} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{Y}^{(r-1)r[\tau_{\max}]} \\ \tilde{Y}^{r1[\tau_{\max}]} & \cdots & \tilde{Y}^{r(r-1)[\tau_{\max}]} & \tilde{Y}^{rr[\tau_{\max}]} \end{bmatrix} > 0, \tag{4.15}$$

for appropriately defined matrices

$$\tilde{Y}^{ab[\tau_{\max}]} = (\tilde{Y}^{ba[\tau_{\max}]})^{\mathrm{T}} \in \Re^{(2r^{2\tau_{\max}+1} n_x) \times (2r^{2\tau_{\max}+1} n_x)},$$
$$\tilde{X}_{ab}^{\mathbf{k}[\tau_{\max}]} = (\tilde{X}_{ba}^{\mathbf{k}[\tau_{\max}]})^{\mathrm{T}} \in \Re^{(2r^{2\tau_{\max}} n_x) \times (2r^{2\tau_{\max}} n_x)},$$

if

$$\tilde{Y}_{\mathbf{i}}^{\mathbf{k}[\tau_{\max}]} > \sum_{\mathbf{j} \in \mathcal{P}(\mathbf{i})} \tilde{X}_{\mathbf{j}}^{\mathbf{k}[\tau_{\max}]}, \quad \forall \mathbf{i}, \mathbf{k} \in \mathbb{I}_2^+, \tag{4.16}$$

$$\begin{bmatrix} \tilde{X}_{11}^{\mathbf{k}[\tau_{\max}]} & \tilde{X}_{12}^{\mathbf{k}[\tau_{\max}]} & \cdots & \tilde{X}_{1r}^{\mathbf{k}[\tau_{\max}]} \\ \tilde{X}_{21}^{\mathbf{k}[\tau_{\max}]} & \tilde{X}_{22}^{\mathbf{k}[\tau_{\max}]} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \tilde{X}_{(r-1)r}^{\mathbf{k}[\tau_{\max}]} \\ \tilde{X}_{r1}^{\mathbf{k}[\tau_{\max}]} & \cdots & \tilde{X}_{r(r-1)}^{\mathbf{k}[\tau_{\max}]} & \tilde{X}_{rr}^{\mathbf{k}[\tau_{\max}]} \end{bmatrix} > \sum_{\mathbf{l} \in \mathcal{P}(\mathbf{k})} \tilde{Y}^{\mathbf{l}[\tau_{\max}]}, \quad \forall \mathbf{k} \in \mathbb{I}_2^+. \tag{4.17}$$

It is noted that for simplicity we have chosen $m = n$ in Algorithm 1. The result with $m \neq n$ can be similarly obtained (with more complex notations). By applying Algorithm 1, a sufficient condition for $\Xi > 0$ is (when $\tau_{\max} \geq 1$)

- for $n = 2\tau_{\max} + 1$: Eqs. (4.11) and (4.12) for $\tau = 0, 1, \ldots, \tau_{\max} - 1$, with $d_\tau = n - 2\tau$, and Eq. (4.14);
- for $n = 2\tau_{\max} + 2$: Eqs. (4.11) and (4.12) for $\tau = 0, 1, \ldots, \tau_{\max} - 1$, with $d_\tau = n - 2\tau$, and Eqs. (4.15)–(4.17).

Example 6. Consider $n=4$, $n \geq \max\{q,p\}$, $\min\{q,p\} \geq 2$ and $r=2$. Then the conditions in Algorithm 1 are expanded as (a)–(b) in Example 5 and below

(c)
$$\begin{bmatrix} \tilde{Y}_{11}^{1111} & \tilde{Y}_{11}^{1112} \\ \tilde{Y}_{11}^{1121} & \tilde{Y}_{11}^{1122} \end{bmatrix} > \tilde{X}_{11}^{11}, \quad \begin{bmatrix} \tilde{Y}_{11}^{1211} & \tilde{Y}_{11}^{1212} \\ \tilde{Y}_{11}^{1221} & \tilde{Y}_{11}^{1222} \end{bmatrix} > \tilde{X}_{11}^{12}, \quad \begin{bmatrix} \tilde{Y}_{11}^{2211} & \tilde{Y}_{11}^{2212} \\ \tilde{Y}_{11}^{2221} & \tilde{Y}_{11}^{2222} \end{bmatrix} > \tilde{X}_{11}^{22},$$
$$\begin{bmatrix} \tilde{Y}_{12}^{1111} & \tilde{Y}_{12}^{1112} \\ \tilde{Y}_{12}^{1121} & \tilde{Y}_{12}^{1122} \end{bmatrix} > \tilde{X}_{12}^{11} + \tilde{X}_{21}^{11}, \quad \begin{bmatrix} \tilde{Y}_{12}^{1211} & \tilde{Y}_{12}^{1212} \\ \tilde{Y}_{12}^{1221} & \tilde{Y}_{12}^{1222} \end{bmatrix} > \tilde{X}_{12}^{12} + \tilde{X}_{21}^{12},$$
$$\begin{bmatrix} \tilde{Y}_{12}^{2211} & \tilde{Y}_{12}^{2212} \\ \tilde{Y}_{12}^{2221} & \tilde{Y}_{12}^{2222} \end{bmatrix} > \tilde{X}_{12}^{22} + \tilde{X}_{21}^{22},$$
$$\begin{bmatrix} \tilde{Y}_{22}^{1111} & \tilde{Y}_{22}^{1112} \\ \tilde{Y}_{22}^{1121} & \tilde{Y}_{22}^{1122} \end{bmatrix} > \tilde{X}_{22}^{11}, \quad \begin{bmatrix} \tilde{Y}_{22}^{1211} & \tilde{Y}_{22}^{1212} \\ \tilde{Y}_{22}^{1221} & \tilde{Y}_{22}^{1222} \end{bmatrix} > \tilde{X}_{22}^{12}, \quad \begin{bmatrix} \tilde{Y}_{22}^{2211} & \tilde{Y}_{22}^{2212} \\ \tilde{Y}_{22}^{2221} & \tilde{Y}_{22}^{2222} \end{bmatrix} > \tilde{X}_{22}^{22};$$

(d)
$$\begin{bmatrix} \tilde{X}_{11}^{11} & \tilde{X}_{12}^{11} \\ \tilde{X}_{21}^{11} & \tilde{X}_{22}^{11} \end{bmatrix} > \tilde{Y}^{11}, \quad \begin{bmatrix} \tilde{X}_{11}^{12} & \tilde{X}_{12}^{12} \\ \tilde{X}_{21}^{12} & \tilde{X}_{22}^{12} \end{bmatrix} > \tilde{Y}^{12} + \tilde{Y}^{21}, \quad \begin{bmatrix} \tilde{X}_{11}^{22} & \tilde{X}_{12}^{22} \\ \tilde{X}_{21}^{22} & \tilde{X}_{22}^{22} \end{bmatrix} > \tilde{Y}^{22};$$

(e)
$$\begin{bmatrix} \tilde{Y}^{11} & \tilde{Y}^{12} \\ \tilde{Y}^{21} & \tilde{Y}^{22} \end{bmatrix} > 0.$$

In the case of Lemma 2 with $q=p=3$, the decision variables for (a)–(e) are (A)–(C) of Example 5 and

(D) symmetric $\tilde{X}_{11}^{11}$, $\tilde{X}_{11}^{12}$, $\tilde{X}_{11}^{22}$, $\tilde{X}_{22}^{11}$, $\tilde{X}_{22}^{12}$, $\tilde{X}_{22}^{22}$, any $\tilde{X}_{12}^{11} = (\tilde{X}_{21}^{11})^{\mathrm{T}}$, $\tilde{X}_{12}^{12} = (\tilde{X}_{21}^{12})^{\mathrm{T}}$, $\tilde{X}_{12}^{22} = (\tilde{X}_{21}^{22})^{\mathrm{T}}$;
(E) symmetric $\tilde{Y}^{11}$, $\tilde{Y}^{22}$, any $\tilde{Y}^{12} = (\tilde{Y}^{21})^{\mathrm{T}}$.

The computational burden for solving the LMI feasibility problem is proportional to $\mathfrak{K}^3\mathfrak{L}$, where $\mathfrak{K}$ is the number of scalar LMI variables and $\mathfrak{L}$ the number of scalar lines in LMIs. Although it is not favorable to give a uniform assessment formula for the computational complexity of the above LMIs in Theorem 2 and Algorithm 1, it is easy to write a tiny software to apply Theorem 2 and Algorithm 1.

5. Numerical Example

5.1. Example 1

Let us consider the following nonlinear model (see [3,14]):

$$x_1(t+1) = x_1(t) - x_1(t)x_2(t) + (5 + x_1(t))u(t),$$
$$x_2(t+1) = -x_1(t) - 0.5x_2(t) + 2x_1(t)u(t).$$

Define $F_1^1(x_1(t)) = (\beta + x_1(t))/(2\beta)$ and $F_1^2(x_1(t)) = (\beta - x_1(t))/(2\beta)$ with $\beta > 0$. When $x_1(t) \in [-\beta, \beta]$, $x_1(t) = \beta F_1^1(x_1(t)) - \beta F_1^2(x_1(t))$. In this way, for $x_1(t) \in [-\beta, \beta]$ the nonlinear model can be exactly represented by the following two rules of discrete-time T–S fuzzy model ($h_1(z(t)) = F_1^1(x_1(t))$, $h_2(z(t)) = F_1^2(x_1(t))$):

Rule 1: If $x_1(t)$ is β then $x(t+1) = \begin{bmatrix} 1 & -\beta \\ -1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 5+\beta \\ 2\beta \end{bmatrix} u(t).$

Rule 2: If $x_1(t)$ is $-\beta$ then $x(t+1) = \begin{bmatrix} 1 & \beta \\ -1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 5-\beta \\ -2\beta \end{bmatrix} u(t).$

For this T–S model, we apply the LMI toolbox in Matlab2006a (the LMI solver is FEASP), and consider the results of three cases.

5.1.1. Case 1: results without using slack matrices

Based on Lemma 1, with Theorem 1 being applied

- for $p=2,\ q=1,\ m=1,\ n=2$, a solution is available for all $\beta \le 1.48$;
- for $p=2,\ q=1,\ m=1,\ n=3$, a solution is available for all $\beta \le 1.62$;
- for $p=2,\ q=1,\ m=1,\ n=4$, a solution is available for all $\beta \le 1.64$;
- for $p=3,\ q=2,\ m=2,\ n=3$, a solution is available for all $\beta \le 1.66$;
- for $p=3,\ q=2,\ m=2,\ n=4$, a solution is available for all $\beta \le 1.67$;
- for $p=3,\ q=2,\ m=3,\ n=4$, a solution is available for all $\beta \le 1.68$.

Note that, when $q=1$, there is no necessity to increase m, i.e., increasing m does not reduce conservativeness. Use of Lemma 2 can give better results, e.g., based on Lemma 2, with Theorem 1 being applied, for $p=3,\ q=3,\ m=4,\ n=4$, a solution is available for all $\beta \le 1.7078$. Consider the case $p=3,\ q=2,\ m=3,\ n=4,\ \beta=1.68$, by applying Lemma 1 and Theorem 1 the following result can be obtained

$$\begin{aligned}
&\tilde{Y}_{11} = [0.3661, -0.8971],\\
&\tilde{Y}_{12} = [1.0283, -1.5603],\\
&\tilde{Y}_{22} = [0.2719, 2.2916],\\
&\tilde{G}_{11} = \begin{bmatrix} 2.4637 & 0.0323 \\ 0.0323 & 5.6501 \end{bmatrix}, \quad \tilde{G}_{12} = \begin{bmatrix} 4.8320 & -2.7823 \\ -2.7823 & 18.7374 \end{bmatrix}, \quad \tilde{G}_{22} = \begin{bmatrix} 2.4893 & 0.0117 \\ 0.0117 & 5.6309 \end{bmatrix}.
\end{aligned}$$

It takes less than 0.3 s to obtain this solution. Choose $x(0) = [1.68, 2.49]^{\mathrm{T}}$. The resulting state responses are shown in Fig. 1 with solid (x_1) and dashed (x_2) lines. We have chosen largest admissible $x_2(0)$, with $x_1(t) \in [-1.68, 1.68],\ \forall t \ge 0$ satisfied.

5.1.2. Case 2: results by Theorem 2 and Lemma 2

Based on Lemma 2, with Eq. (4.7), a solution is available for all $\beta \le 1.7415$. Based on Lemma 2, with Eq. (4.8), a solution is available for all $\beta \le 1.8106$. With "Theorem 1" of [4], a solution is available for all $\beta \le 1.7669$. For $\beta = 1.8106$, by applying Lemma 2 and Eq. (4.8) the following result can be obtained

$$\begin{aligned}
&\tilde{Y}_{11} = [-0.0014334, -0.0202916], \quad \tilde{Y}_{12} = [0.0014444, -0.0033930],\\
&\tilde{Y}_{22} = [0.0114151, 0.0060006],
\end{aligned}$$

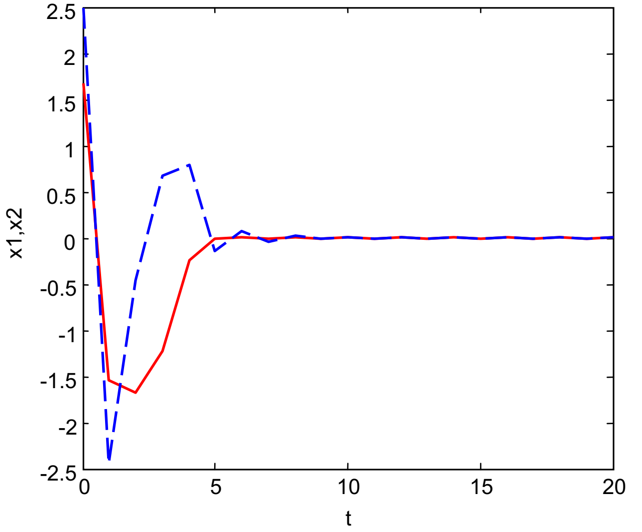

Fig. 1. State responses of the closed-loop system.

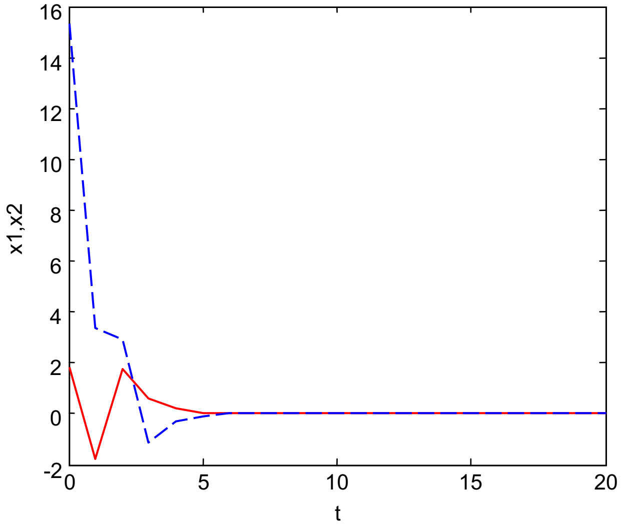

Fig. 2. State responses of the closed-loop system.

$$\tilde{G}_{11} = \begin{bmatrix} 0.0189548 & 0.0313122 \\ 0.0102716 & 0.0924549 \end{bmatrix}, \quad \tilde{G}_{12} = \begin{bmatrix} -0.0200112 & -0.1006179 \\ -0.0002051 & 0.0855013 \end{bmatrix},$$

$$\tilde{G}_{22} = \begin{bmatrix} 0.0888121 & -0.0103304 \\ -0.0385826 & 0.0232637 \end{bmatrix}.$$

It takes about $3\frac{1}{3}$ min to obtain this solution. Choose $x(0) = [1.8106, 15.3]^T$. The resultant state responses are shown in Fig. 2 with solid (x_1) and dashed (x_2) lines. We have chosen the largest admissible $x_2(0)$, with $x_1(t) \in [-1.8106, 1.8106]$, $\forall t \geq 0$ satisfied.

5.1.3. Case 3: results by Algorithm 1 and Lemma 2

Based on Lemma 2, with Algorithm 1 applied, a solution is available for all $\beta \leq 1.8106$. For $\beta = 1.8106$, by applying Lemma 2 and Algorithm 1, the following result can be obtained:

$$\tilde{Y}_{11} = [-0.0000187, -0.0002627], \quad \tilde{Y}_{12} = [0.0000193, -0.0000453]$$
$$\tilde{Y}_{22} = [0.0001480, 0.0000778],$$
$$\tilde{G}_{11} = \begin{bmatrix} 0.0002456 & 0.0004054 \\ 0.0001337 & 0.0011972 \end{bmatrix}, \quad \tilde{G}_{12} = \begin{bmatrix} -0.0002442 & -0.0013364 \\ -0.0000014 & 0.0011053 \end{bmatrix},$$
$$\tilde{G}_{22} = \begin{bmatrix} 0.0011516 & -0.0001338 \\ -0.0005002 & 0.0003015 \end{bmatrix}.$$

It takes about $1\frac{4}{5}$ h to obtain this solution. Choose $x(0) = [1.8106, 15.3]^T$. The resulting state responses are nearly identical to those in Fig. 2, which are omitted here. We have chosen the largest admissible $x_2(0)$, with $x_1(t) \in [-1.8106, 1.8106]$, $\forall t \geq 0$ satisfied. For this example, Algorithm 1 has not achieved better results than Theorem 2, i.e., fully exploring the slack matrix technique in this example does not bring any advantage.

The computational complexity with respect to $\{p, q, m, n\}$ and slack matrix variables can be analyzed similarly as in [25]. As having been reported in [25], introducing slack matrix variables can be more effective (than increasing $n-p$ and $m-q$) for reducing the conservatism, but the computational burden can be increased considerably. It is reported in [14] that the nonquadratic Lyapunov method can be considerably less conservative than the common quadratic one. Hence, at least, the stability conclusions in this paper can be applied when the procedures in [25] fail.

5.2. Example 2

Let us consider the nonlinear model of a continuous stirred tank reactor, as in [3]. Let us omit the model details. The readers should refer to [3]. By applying “Theorem 2”, “Theorem 4” and “Theorem 6” of [3], the stabilization conditions are satisfied for all $\beta \leq 53.00$, $\beta \leq 54.04$ and $\beta \leq 54.08$, respectively. Here, the larger the upper bound of β, the less conservative the stabilization conditions. Based on Lemma 2, with Algorithm 1 applied, a solution is available for all $\beta \leq 54.32$. It takes about $3\frac{2}{3}$ h to obtain this solution. Clearly, the feasible region of β has been improved.

6. Conclusions

The asymptotically necessary and sufficient (ANS) stability conditions in the sense of HPNQ Lyapunov functions are obtained for the discrete-time T–S fuzzy models. Various results can be obtained by increasing the complexities of Lyapunov functions, of the feedback laws, and of the slack matrices. For a specific selection of Lyapunov function and a specific selection of the form of feedback law, less conservative sufficient stability conditions can be obtained, by increasing the complexity parameters $\{m, n\}$. This represents a parallel result of [8] based on [7], but at the same time applies the slack matrices. It would be promising to incorporate the results in this paper into the H_∞ control and networked control, for which the techniques in [22–24] could be applied.

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