ANALYSIS AND SYNTHESIS OF FUZZY STOCHASTIC SYSTEMS VIA LMI APPROACH

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Abstract: This paper studied a general class of nonlinear stochastic systems, which can be approximated by T-S fuzzy model. The problem both the control and the state entering into the diffusion term is discussed here. A PDC state feedback controller is proposed to guarantee the mean-square stability of the closed-loop system. All the results are represented by the linear matrix inequalities.

Key words: Fuzzy model linear matrix inequalities stability analysis stochastic systems

1. Introduction

The stochastic dynamical system described by Ito stochastic differential equations has been investigated for the passed four decades and the linear stochastic system theory has been well developed and established. (See [1]-[4] and the references therein). As for the nonlinear case, since the analysis and synthesis of the nonlinear determined systems are not easy, the nonlinear stochastic systems are more difficult to deal with[5].

Recently, dynamic Takagi-Sugeno fuzzy controllers have been successfully applied to the stabilization control design of nonlinear systems[6]. The T-S fuzzy model has been proved to be a very good representation for a certain class of nonlinear dynamic systems[7]. The corresponding parallel distributed compensation (PDC) has been widely studied and is proved to be a very appealing approach due to the powerful tool, linear matrix inequalities (LMI's), which can be efficiently solved by the interior point algorithm[8]. However, there are very few studies concerning with stochastic systems[9], especially for the problem both the control and the state entering into the diffusion term, to the best of our knowledge, there is no any works on this problem.

In this paper, a given nonlinear stochastic system is represented by a Takagi-Sugeno stochastic model. The stability and design problems are solved using the LMI's. On the other hand, this work is obviously different from [9], where the diffusion term is not considered.

2. Stability Analysis for Fuzzy Stochastic Model

Let (Ω, F, P, F_t) be a given filtered probability space with a standard one-dimensional Brownian motion $\omega(t)$ on $[0, +\infty]$ (with $\omega(0) = 0$). We define the following Hilbert space[3]:

$$L_F^2(\Re^k) = \left\{ \begin{array}{l} \phi(\cdot):[0,+\infty]\times\Omega \rightarrow \Re^k \mid \\ \phi(\cdot) \text{ is } F_t \text{ - adapted, measurable,} \\ \text{and } E\int |\phi(t,\omega)|^2\, dt < +\infty \end{array} \right\}$$

with the norm $\|\phi(\cdot)\| = [E\int_0^{+\infty} |\phi(t,\omega)|^2\, dt]^{1/2}$.

Specifically, the T-S fuzzy stochastic system is described by fuzzy IF-THEN rules, which locally represent linear input-output relations of nonlinear stochastic systems. The ith rule of the fuzzy model is of the following Ito differential dynamic equation.

Plant Rule i :

IF $\xi_1(t)$ is μ_{i1} and ... and $\xi_p(t)$ is μ_{ip}

THEN

$$dx(t) = [A_i x(t) + B_i u(t)]\,dt + [C_i x(t) + D_i u(t)]\,d\omega(t)$$

$$x(0) = x_0 \in \Re^n, \qquad i = 1, 2, \ldots, r \qquad (2.1)$$

where μ_{ij} is the fuzzy set, $x(t) \in \Re^n$ is the state vector, $u(t) \in L_F^2(\Re^m)$ is the input vector. A_i B_i C_i D_i are some constant matrices of compatible dimensions that describe the system. r is the number of IF-THEN rules, and $\xi(t) = [\xi_1(t)\ \ \xi_2(t) \cdots \xi_p(t)]$ are the precise variables.

Given a pair of $(x(t), u(t))$, the final outputs of the fuzzy stochastic system are inferred as follows:

$$dx(t) = [Ax(t) + Bu(t)]\,dt + [Cx(t) + Du(t)]\,d\omega(t) \quad (2.2)$$

where $A = \sum_{i=1}^{r} h_i(\xi(t))A_i \qquad B = \sum_{i=1}^{r} h_i(\xi(t))B_i$

$$C = \sum_{i=1}^{r} h_i(\xi(t))C_i \qquad D = \sum_{i=1}^{r} h_i(\xi(t))D_i \qquad (2.3)$$

$$h_i(\xi(t)) = \frac{w_i(\xi(t))}{\sum_{j=1}^{r} w_j(\xi(t))}, \qquad w_i(\xi(t)) = \prod_{j=1}^{p} \mu_{ij}(\xi(t))$$

To go on our research, some ordinary assumptions have to be made:

Assumption 1: $h_i(\xi(t)) \geq 0$, for $i = 1, 2, \ldots, r$

Assumption 2: $\sum_{i=1}^{r} h_i(\xi(t)) = 1$ for all t

Assumption 3: all membership function $\mu_{ij}(\cdot)$ are continuous and piecewise continuously differentiable and the deffuzification method is also continuous.

Definition 1: the autonomous fuzzy stochastic system (2.2) with $u(t) \equiv 0$ is called **mean-square stable** (with respected to x_0) if the corresponding state $x(t)$ of (2.2) with the initial state x_0 satisfies $\lim_{t \to +\infty} E[x(t)x^T(t)] = 0$.

Definition 2: the fuzzy stochastic system (2.2) is called **mean-square stabilizable** (with respected to x_0) if there exists a mean-square feedback control law $u(\cdot)$ satisfies $\lim_{t \to +\infty} E[(x(t)x^T(t)] = 0$.

A sufficient condition that guarantee the mean-square stability of the fuzzy stochastic is obtain in terms of Lyapunov's direct method.

Theorem 1: The fuzzy stochastic system described by (2.2) is mean-square stable if there exists a common matrix $P > 0$ such that

$$\begin{bmatrix} A_i^T P + PA_i & C_i^T P \\ PC_i & -P \end{bmatrix} < 0, \quad \text{for } i = 1, 2, \ldots, r. \quad (2.4)$$

Proof: let $M(t) = E[x(t)x^T(t)]$, then by Ito's formula, we can get:

$$\dot{M}(t) = AM(t) + M(t)A^T + CM(t)C^T \qquad (2.5)$$

It is evident that $M(t) \geq 0$, So a Lyapunov function candidate can be selected as:

$$V(M(t)) = Trace[M(t)P] \qquad (2.6)$$

The derivative of the Lyapunov function along the trajectories of (2.2) is

$$\dot{V}(M(t)) = Trace[\dot{M}(t)P]$$
$$= Trace[AM(t)P + M(t)A^T P + CM(t)C^T P]$$
$$= Trace[M(t)(A^T P + PA + C^T PC)] \qquad (2.7)$$

Obviously, if the matrix inequalities in (2.4) hold then

$$\sum_{i=1}^{r} h_i(\xi(t)) \begin{bmatrix} A_i^T P + PA_i & C_i^T P \\ PC_i & -P \end{bmatrix} < 0 \qquad (2.8)$$

that is:

$$\begin{bmatrix} A^T P + PA & C^T P \\ PC & -P \end{bmatrix} < 0 \qquad (2.9)$$

From the Schur's Lemma, we can get $\dot{V}(M(t)) < 0$. It means that the equilibrium of the dynamic system described by (2.5) is asymptotically stable in the large, i.e.

$\lim_{t \to +\infty} M(t) = 0$. So from definition 1, the fuzzy stochastic system (2.2) is mean-square stable. □

3. State Feedback Fuzzy Controller Design

Based on the parallel distributed compensation[6], we consider the following fuzzy control law

Regulator Rule i :

IF $\xi_1(t)$ is μ_{i1} and … and $\xi_p(t)$ is μ_{ip}

THEN $u(t) = K_i x(t) \qquad i = 1, 2, \ldots, r. \qquad (3.1)$

The overall state feedback fuzzy control law is represented by

$$u(t) = Kx(t) \qquad (3.2)$$

where
$$K = \sum_{i=1}^{r} h_i(\xi(t)) K_i \qquad (3.3)$$

The design of state feedback fuzzy controller is to determine the local feedback gains K_i such that the following closed-loop system (3.4) is asymptotically stable

$$dx(t) = \sum_{i,j=1}^{r} h_i(\xi(t)) h_j(\xi(t)) \{ [A_i + B_i K_j] x(t)\, dt + [C_i + D_i K_j] x(t)\, d\omega(t) \} \qquad (3.4)$$

Theorem 2: There exists a state feedback fuzzy control law (3.2) such that the closed-loop system (3.4) is mean-square stable if there exist matrices $X > 0$ and Y_i satisfying the following LMI's:

$$\begin{bmatrix} A_i X + X A_i^T + B_i Y_i + Y_i^T B_i^T & X C_i^T + Y_i^T D_i^T \\ C_i X + D_i Y_i & -X \end{bmatrix} < 0 \qquad (3.5)$$

$$\begin{bmatrix} G_{ij} & \Pi_{ij}^T \\ \Pi_{ij} & -2X \end{bmatrix} \le 0, \qquad i < j \qquad (3.6)$$

where

$$G_{ij} = A_i X + X A_i^T + A_j X + X A_j^T + B_i Y_j + Y_j^T B_i^T + B_j Y_i + Y_i^T B_j^T$$

$$\Pi_{ij} = C_i X + D_i Y_j + C_j X + D_j Y_i \qquad (3.7)$$

for all i and j excepting the pairs (i, j) such that $h_i(\xi(t)) h_j(\xi(t)) = 0,\ \forall t$. And the state feedback gain can be constructed as:

$$K_i = Y_i X^{-1}, \quad i = 1, 2, \ldots, r. \qquad (3.8)$$

Proof: let $M(t) = E[x(t)x^T(t)]$, then by Ito's formula, we can get:

$$\dot{M}(t) = (A + BK)M(t) + M(t)(A + BK)^T + (C + DK)M(t)(C + DK)^T \qquad (3.9)$$

Select the Lyapunov function as

$$V(M(t)) = Trace[M(t)P] \qquad (3.10)$$

where $P = X^{-1} > 0$, then

$$\dot{V}(M(t)) = Trace[\dot{M}(t)P] = Trace\{M(t)[P(A + BK) + (A + BK)^T P + (C + DK)^T P(C + DK)\} \qquad (3.11)$$

From the LMI's in (3.5) and (3.6), we can get:

$$\sum_{i,j=1}^{r} h_i(\xi(t)) h_j(\xi(t)) \begin{bmatrix} A_i X + X A_i^T + B_i Y_j + Y_j^T B_i^T & X C_i^T + Y_j^T D_i^T \\ C_i X + D_i Y_j & -X \end{bmatrix} < 0 \qquad (3.12)$$

It is equivalent to

$$\begin{bmatrix} AX + XA^T + BY + Y^T B^T & XC^T + Y^T D^T \\ CX + DY & -X \end{bmatrix} < 0 \qquad (3.13)$$

where
$$Y = \sum_{i=1}^{r} h_i(\xi(t)) Y_i \qquad (3.14)$$

that is: $$AX + XA^T + BY + Y^T B^T + (CX + DY)^T X^{-1} (CX + DY) < 0 \qquad (3.15)$$

Select K_i as (3.8), then

$$K = \sum_{i=1}^{r} h_i(\xi(t)) K_i = \sum_{i=1}^{r} h_i(\xi(t)) Y_i X^{-1} = Y X^{-1} \qquad (3.16)$$

Furthermore, premultiplying and postmultiplying to (3.15) by positive-definite matrix P, we can get

$$P(A + BK) + (A + BK)^T P$$

$$+(C+DK)^T P(C+DK) < 0 \qquad (3.17)$$

from (3.11), $\dot{V}(M(t)) < 0$. It means that the equilibrium of the dynamic system described by (3.9) is asymptotically stable in the large, i.e. $\lim_{t \to +\infty} M(t) = 0$. So the stochastic system (3.4) is mean-square stable. □

4. Conclusions

In this paper, a novel approach is proposed to study the nonlinear stochastic systems, which is based on the T-S modeling and the PDC design. The problem of mean-square stability is investigated and the mean-square stabilizing control law is proposed. Under the framework of T-S fuzzy stochastic systems proposed here , more works, such as stochastic observer, stochastic output feedback, stochastic H_2 control, stochastic H_∞ control, etc, can be discussed in detail. Some of them are in progress. Another possible direction for future work is to investigate the fuzzy stochastic systems with time-delay.

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