Superconvergence of Finite Element Approximations to Parabolic and Hyperbolic Integro-Differential Equations*

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Abstract: The object of this paper is to investigate the superconvergence properties of finite element approximations to parabolic and hyperbolic integro-differential equations. The quasi projection technique introduced earlier by Douglas et al. is developed to derive the $O(h^{2r})$ order knot superconvergence in the case of a single space variable, and to show the optimal order negative norm estimates in the case of several space variables.

Key words: superconvergence, parabolic and hyperbolic integro-differential equation, finite element


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1 Introduction

Recently, considerable attention has been devoted to the finite element analysis for partial integro-differential equations, see, for example, Yanik and Fairweather\cite{1}, Cannon and Lin\cite{2,3}, Chen and Shih\cite{4}, Lin, Thomée and Wahlbin\cite{5}, Thomée and Zhang\cite{6} and Zhang\cite{7,8}. The main tool used for this kind of equations is the Ritz-Volterra projection \cite{5,7} as against Ritz projection for parabolic equation. In this paper, we are concerned primarily with the analysis of knot superconvergence and negative norm estimates associated with the finite element solutions of second order liner parabolic and hyperbolic integro-differential equations. It is well known that \cite{9,10} the finite element solutions for elliptic and parabolic problems converge with an error order that is of $O(h^{r+1})$ in $L_\infty$ norm at best globally when the approximation space consists of piecewise polynomial functions of degree $r$. However, at each mesh node, the solutions admit an $O(h^{2r})$ order superconvergence in the case of a single space variable. In this paper, based on a sequence of Ritz-Volterra projections, the quasi projection technique introduced earlier by Douglas, Dupont and Wheeler\cite{16} for parabolic

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equation is developed to show that thus superconvergence result also holds for parabolic and hyperbolic integro-differential equations. Moreover, the optimal order negative norm estimates are also established in the case of several space variables. It should be indicated that some similar results seem to have been obtained in article [11], but the proof is wrong because the functional $F(\eta)$ introduced in Theorem 4.1 does not satisfy the condition in Lemma 4.1 with the desired bound (see [11]). Thus, our demonstration also corrects the mistakes in article [11].

Consider the following parabolic integro-differential equation:

$$
\begin{align*}
    u_t + A(t)u &= \int_0^t B(t, \tau)u(\tau) \, d\tau + f(t, x), \quad \text{in } \Omega \times J, \\
    u &= 0, \quad \text{on } \partial \Omega \times J, \\
    u(0) &= u_0(x), \quad \text{in } \Omega,
\end{align*}
$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($1 \leq n \leq 3$) with smooth boundary $\partial \Omega$, $J = (0, T]$, $T < \infty$, $u_t = \partial u/\partial t$, $f$ and $u_0$ are known functions. Furthermore, $A(t)$ is a selfadjoint, uniformly positive definite, second order elliptic partial differential operator of the form

$$
A(t) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (a_{ij}(t, x) \frac{\partial}{\partial x_i}) + a_0(t, x)I,
$$

and $B(t, \tau)$ is a general second order partial differential operator of the form

$$
B(t, \tau) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} (b_{ij}(t, \tau, x) \frac{\partial}{\partial x_i}) + \sum_{i=1}^n b_i(t, \tau, x) \frac{\partial}{\partial x_i} + b_0(t, \tau, x)I.
$$

Assume that the coefficients of $A(t)$ and $B(t, \tau)$ are sufficiently smooth in our argument.

Let $H^s(\Omega)$ be the usual Sobolev space with the norm $\| \cdot \|_s$. The normed dual of $H^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. $H^0_0(\Omega)$ means the completion of $C_0^\infty(\Omega)$ under the norm $\| \cdot \|_1$. The notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ represent the inner product and induced norm on $L_2(\Omega)$.

Let $S_h(0 < h < 1)$ be a family of finite dimensional subspaces of $H^1_0(\Omega)$ with the following approximation property: for a given integer $r \geq 1$

$$
\inf_{\chi \in S_h} \{ \| v - \chi \| + h \| v - \chi \|_1 \} \leq Ch^s \| v \|_{s}, \quad v \in H^s(\Omega) \bigcap H^1_0(\Omega), \quad 1 \leq s \leq r + 1.
$$

(1.2)

Now the semidiscrete finite element approximation to the problem (1.1) is defined as finding $u_h(t) : J \to S_h$ such that

$$
\begin{align*}
    (u_h, \varphi) + A(t; u_h, \varphi) &= \int_0^t B(t, \tau; u(\tau), \varphi) \, d\tau + (f, \varphi), \quad \varphi \in S_h, \\
    u_h(0) &\in S_h,
\end{align*}
$$

(1.3)

where $A(t; u, v)$ and $B(t, \tau; u, v)$ are bilinear forms on $H^1_0(\Omega) \times H^1_0(\Omega)$ associated with the operators $A(t)$ and $B(t, \tau)$, respectively.

This paper is organized as follows: In Section 2 we develop the quasi projection technique based on the Ritz-Volterra projection and establish some related estimates and optimal negative norm estimates. Section 3 is devoted to derive the knot superconvergence in the case of one space variable. Finally, in Section 4, we treat hyperbolic integro-differential equation and obtain analogous results as those in Sections 2 and 3.
2 Quasi Projection and Related Estimates

Quasi projection technique plays a crucial role in our analysis. Originally it is introduced by Douglas et al. for parabolic equations. In this section, we develop this technique so that it is applicable for parabolic integro-differential equations. First, define the Ritz-Volterra projection \( V_h : C(\bar{J}; H_0^1(\Omega)) \to \bar{C}(\bar{J}; S_h) \) such that

\[
A(t; u - V_h u, \chi) = \int_0^t B(t, \tau; u(\tau) - V_h u(\tau), \chi) \, d\tau, \quad \forall \chi \in S_h, \quad t \in J. \tag{2.1}
\]

Obviously, the Ritz-Volterra projection \( V_h \) is a natural generalization of the Ritz projection \( R_h \). When \( B(t, \tau) \equiv 0, \, V_h = R_h \) holds. Let \( \rho_0 = V_h u - u \). Now the quasi projections are defined as maps \( \rho_j : J \to S_h \) with \( j \geq 1 \) such that

\[
A(t; \rho_j, \chi) = \int_0^t B(t, \tau; \rho_j(\tau), \chi) \, d\tau - (\frac{\partial \rho_{j-1}}{\partial t}, \chi), \, \chi \in S_h. \tag{2.2}
\]

Set \( \theta_0 = u_h - V_h u, \, \theta_j = u_h - V_h u - \rho_1 - \cdots - \rho_j, \, j \geq 1 \). Then the error \( u_h - u \) can be decomposed as

\[
u_h - u = \theta_0 + \rho_0 = \theta_j + \rho_0 + \rho_1 + \cdots + \rho_j \tag{2.3}\]

Thus, to estimate the error \( u_h - u \) we only need to estimate \( \theta_j \) and \( \rho_j \) \( (j \geq 0) \). This is why we introduce the quasi projections \( \rho_j \). Below we give some estimates for \( \rho_j \) and \( \theta_j \).

**Lemma 2.1** Let \( \phi \in H_0^1(\Omega) \) satisfy

\[
A(t; \phi, \chi) = F(\chi), \quad \forall \chi \in S_h,
\]

where \( F : H_0^1(\Omega) \to R \) is a linear functional with the property

\[
|F(\psi)| \leq M_p \|\psi\|_p, \quad \forall \psi \in H^p(\Omega) \cap H_0^1(\Omega), \quad p = 1, 2, \ldots, r + 1.
\]

Then, for \( s = -1, 0, \ldots, r - 1 \), we have

\[
\|\phi\|_{-s} \leq C [(M_1 + \inf_{\chi \in S_h} \|\phi - \chi\|_1) h^{s+1} + M_{s+2}].
\]

**Proof.** For \( g \in H^s(\Omega) \), let \( \psi \in H^{s+2}(\Omega) \cap H_0^1(\Omega) \) satisfy

\[
A(t; \psi) = g, \quad \text{in } \Omega; \quad \|\psi\|_{s+2} \leq C \|g\|_s, \quad s \geq 0.
\]

Note that \( R_h \psi \in S_h \) is the Ritz projection of \( \psi \). For \( \chi \in S_h \), we have

\[
(\phi, g) = A(t; \phi, \psi) = A(t; \phi, \psi - R_h \psi) + F(R_h \psi)
= A(t; \phi - \chi, \psi - R_h \psi) - F(\psi - R_h \psi) + F(\psi)
\leq C \|\phi - \chi\|_1 \|\psi - R_h \psi\|_1 + M_1 \|\psi - R_h \psi\|_1 + M_{s+2} \|\psi\|_{s+2}
\leq C [(\|\phi - \chi\|_1 + M_1) h^{s+1} + M_{s+2}] \|\psi\|_{s+2}, \quad -1 \leq s \leq r - 1.
\]

The proof is completed.

**Lemma 2.2** Let \( k \geq 0, \, 1 \leq q \leq r + 1, \, -1 \leq s \leq r - 1 \). Then, for \( \rho_0 = V_h u - u \), we have

\[
\|\frac{\partial^k \rho_0}{\partial t^k}\|_{-s} \leq C h^{q+s} \left( \sum_{i=0}^{k} \|\frac{\partial^i u}{\partial t^i}\|_q + \int_0^t \|u\|_q \, d\tau \right).
\]
Proof. In Lemma 2.1, take $\phi = \rho_0$. \[ F(\psi) = \int_0^t B(t, \tau; \rho_0(\tau), \psi) \, d\tau \] (see (2.1)), and note that
\[
|F(\psi)| \leq C \int_0^t \|\rho_0(\tau)\|_1 \, d\tau \|\psi\|_1 = M_1 \|\psi\|_1,
\]
\[
|F(\psi)| = \int_0^t (\rho_0, B^*(t, \tau) \psi) \, d\tau \leq C \int_0^t \|\rho_0(\tau)\|_{-s} \, d\tau \|\psi\|_{s+2} = M_{s+2} \|\psi\|_{s+2},
\]
where $B^*(t, \tau)$ is the adjoint of $B(t, \tau)$. Hence, by Lemma 2.1 and the approximation property, we have
\[
\|\rho_0\|_{-s} \leq C \theta^{s+1} \left( \int_0^t \|\rho_0\|_1 \, d\tau + h^{q-1} \|u(t)\|_q \right) + C \int_0^t \|\rho_0\|_{-s} \, d\tau.
\]
First, set $s = -1$ and apply Gronwall Lemma to obtain the bound of $\|\rho_0\|_1$, and then the desired estimate for $k = 0$ follows from applying Gronwall Lemma again. For $k = 1$, by differentiating (2.1) with respect to $t$, we obtain
\[
A(t, \frac{\partial \rho_0}{\partial t}, \chi) = -A_t(t, \rho_0, \chi) + B(t, t; \rho_0(t), \chi) + \int_0^t B_t(t, \tau; \rho_0(\tau), \chi) \, d\tau.
\]
Let $F(\chi)$ be given by the right hand side of the last equality. Again using Lemma 2.1 with $\phi = \frac{\partial \rho_0}{\partial t}$ yields
\[
\left\| \frac{\partial \rho_0}{\partial t} \right\|_{-s} \leq C \theta^{s+1} \left( \int_0^t \|\rho_0\|_1 \, d\tau + h^{q-1} \|u(t)\|_q \right) + C \int_0^t \|\rho_0\|_{-s} \, d\tau.
\]
It implies Lemma 2.2 with $k = 1$. The proof is completed by treating $k \geq 2$ inductively, and using further differentiation of (2.1).

**Theorem 2.1** Let $j, k \geq 0$, $1 \leq q \leq r + 1$ and $-1 \leq s + 2j \leq r - 1$. Then
\[
\left\| \frac{\partial^k \rho_j}{\partial t^k} \right\|_{-s} \leq C \theta^{s+2j} \left( \sum_{i=0}^{j+k} \|\frac{\partial^i u}{\partial t^i}\|_q + \sum_{i=0}^{j} \int_0^t \|\frac{\partial^i u}{\partial t^i}\|_q \, d\tau \right).
\]

**Proof.** The proof will be completed by induction on $j$. For $j = 0$, see Lemma 2.2. Suppose that Theorem 2.1 holds for $j - 1$. For $j \geq 1$, according to (2.2), let
\[
F(\psi) = \int_0^t B(t, \tau; \rho_j(\tau), \psi) \, d\tau - (\frac{\partial \rho_{j-1}}{\partial t}, \psi),
\]
and we have
\[
|F(\psi)| \leq C \int_0^t \|\rho_j\|_1 \, d\tau + \|\frac{\partial \rho_{j-1}}{\partial t}\|_{-1} \|\psi\|_1 = M_1 \|\psi\|_1,
\]
\[
|F(\psi)| \leq C \int_0^t \|\rho_j(\tau)\|_{-s} \, d\tau + \|\frac{\partial \rho_{j-1}}{\partial t}\|_{-s-2} \|\psi\|_{s+2} = M_{s+2} \|\psi\|_{s+2}.
\]
Then $F$ satisfies the hypotheses of Lemma 2.1 with $\phi = \rho_j \in S_h$. Therefore we have
\[
\|\rho_j\|_{-s} \leq C \left( \int_0^t \|\rho_j\|_1 \, d\tau + \|\frac{\partial \rho_{j-1}}{\partial t}\|_{-1} \right) \theta^{s+1} + C \left( \int_0^t \|\rho_j\|_{-s} \, d\tau + \|\frac{\partial \rho_{j-1}}{\partial t}\|_{-s-2} \right).
\]
First, take $s = -1$ and apply Gronwall Lemma to obtain the bound of $||\rho_j||_1$, and then apply Gronwall Lemma again to yield

$$
||\rho_j||_{-s} \leq C( ||\frac{\partial \rho_{j-1}}{\partial t}||_{-1} + \int_0^t ||\frac{\partial \rho_{j-1}}{\partial t}||_{-1} \, dt)^{s+1}
+ C( ||\frac{\partial \rho_{j-1}}{\partial t}||_{-s-2} + \int_0^t ||\frac{\partial \rho_{j-1}}{\partial t}||_{-s-2} \, dt).$$

Apply induction on $j$ and note that $s+2+2(j-1) = s+2j \leq r-1$ to obtain the estimate of $\rho_j$. For the time derivatives of $\rho_j$, differentiating (2.2) and following the proof of Lemma 2.2, we derive the desired estimates inductively.

Now consider the estimates of $\theta_j$, $j \geq 0$. First, from the equations (1.1), (1.3) and (2.1) we see that $\theta_0 = u_h - V_h u$ satisfies

$$(\frac{\partial \theta_0}{\partial t}, \chi) + A(t; \theta_0, \chi) = \int_0^t B(t, \tau; \theta_0, \chi) \, d\tau - (\frac{\partial \theta_0}{\partial t}, \chi), \quad \chi \in S_h.$$

Then, a simple argument by using (2.2) shows that $\theta_j = \theta_0 - \rho_1 - \cdots - \rho_j$ satisfies

$$(\frac{\partial \theta_j}{\partial t}, \chi) + A(t; \theta_j, \chi) = \int_0^t B(t, \tau; \theta_j, \chi) \, d\tau - (\frac{\partial \theta_j}{\partial t}, \chi), \quad \chi \in S_h. \quad (2.4)$$

**Theorem 2.2** Assume that the initial value $u_h(0)$ be chosen so that $\theta_j(0) = 0$, that is,

$$u_h(0) = V_h u(0), \quad j = 1, \quad u_h(0) = V_h u(0) + \rho_1(0) + \cdots + \rho_j(0), \quad j \geq 1. \quad (2.5)$$

Then, for $1 \leq q \leq r + 1$, we have

$$||\theta_j||_1 \leq \begin{cases} 
C h^{q+2j+1} \sum_{i=0}^{j+1} \left( \int_0^t \left| \frac{\partial u}{\partial t} \right|^2 \, dt \right)^{\frac{1}{2}}, & 2j \leq r - 1, \\
C h^{q+2j+1} \left[ \sum_{i=0}^{j+1} \left| \frac{\partial u}{\partial t} \right|^2 \right] \left( \int_0^t \left| \frac{\partial u}{\partial t} \right|^2 \, dt \right)^{\frac{1}{2}}, & 2j \leq r - 2.
\end{cases}$$

**Proof.** Taking $\chi = \frac{\partial \theta_j}{\partial t}$ in (2.4) and integrating from 0 to $t$, we obtain

$$\int_0^t \left| \frac{\partial \theta_j}{\partial t} \right|^2 \, dt + \frac{1}{2} A(t; \theta_j, \theta_j) - \frac{1}{2} \int_0^t A_t(\tau; \theta_j(\tau), \theta_j(\tau)) \, d\tau$$

$$= \int_0^t B(t, \tau; \theta_j(\tau), \theta_j(t)) \, d\tau - \int_0^t B(\tau, \tau; \theta_j(\tau), \theta_j(\tau)) \, d\tau$$

$$- \int_0^t \int_0^t B_t(\tau, s; \theta_j(s), \theta_j(\tau)) \, dsd\tau - \int_0^t \left( \frac{\partial \rho_j}{\partial t}, \frac{\partial \theta_j}{\partial t} \right) \, dt. \quad (2.6)$$

It follows from Cauchy inequality that

$$||\theta_j(t)||_1^2 \leq C \int_0^t ||\theta_j(\tau)||_1^2 \, d\tau + C \int_0^t ||\frac{\partial \rho_j}{\partial t}||^2 \, d\tau.$$

Applying Gronwall Lemma and Theorem 2.1 with $s = 0$, $k = 1$, we have

$$||\theta_j(t)||_1 \leq C h^{q+2j} \sum_{i=0}^{j+1} \left( \int_0^t \left| \frac{\partial u}{\partial t} \right|^2 \, dt \right)^{\frac{1}{2}}, \quad 2j \leq r - 1.$$

This establishes the first inequality in Theorem 2.2. If $r$ is odd, the choices $2j = r - 1$ and $q = r + 1$ produce an $O(h^{2r})$ order estimate for $\theta_j$ in $H^1(\Omega)$. However, if $r$ is even, only
an $O(h^{2r-1})$ order estimate can be obtained. We need the following argument to regain the $O(h^{2r})$ order estimate. Integrate the last right hand term in (2.6) by parts in time to obtain

$$\int_0^t (\frac{\partial \rho_j}{\partial t}, \frac{\partial \theta_j}{\partial t}) \, \mathrm{d}t = (\frac{\partial \rho_j}{\partial t}, \theta_j) - \int_0^t (\frac{\partial^2 \rho_j}{\partial t^2}, \theta_j) \, \mathrm{d}t.$$ 

Substituting this into (2.6) and using Cauchy inequality and Gronwall Lemma, we obtain

$$\|\theta_j(t)\|_{L_2}^2 \leq C \left( \int_0^t \|\rho_j\|_{L_2}^2 \, \mathrm{d}t \right)^{2-1} + \int_0^t \|\rho_j\|_{L_2}^2 \, \mathrm{d}t + \int_0^t \frac{\partial^2 \rho_j}{\partial t^2} \|\theta_j\|_{L_2} \, \mathrm{d}t.$$ 

The proof is completed by using Theorem 2.1 with $s = 1, k = 1, 2, 1 + 2j \leq r - 1$.

A direct result of the above estimates is the optimal negative norm estimate of the error $u - u_h$.

Theorem 2.3 Let $u$ and $u_h$ be solutions of the problems (1.1) and (1.3), respectively, $u_h(0)$ be given by (2.5), and $1 \leq q \leq r + 1$. Then

$$\|u - u_h\|_{-s} \leq C h^{s+q} \left\{ \sum_{i=0}^{j+1} \|\frac{\partial^i u}{\partial t^i}\|_{L^q}^q + \sum_{i=0}^{j+2} \left( \int_0^t \|\frac{\partial^i u}{\partial t^i}\|_{L^q}^q \, \mathrm{d}t \right)^{1/2} \right\}, \quad 0 \leq s \leq 2j \leq r - 1,$n

$$\|u - u_h\|_{-s} \leq C h^{s+q} \left\{ \sum_{i=0}^{j+1} \|\frac{\partial^i u}{\partial t^i}\|_{L^q}^q + \sum_{i=0}^{j+2} \left( \int_0^t \|\frac{\partial^i u}{\partial t^i}\|_{L^q}^q \, \mathrm{d}t \right)^{1/2} \right\}, \quad 0 \leq s \leq 2j + 1 \leq r - 1.$$ 

Proof. From (2.3), the proof is completed immediately by using Theorems 2.1 and 2.2 together with the imbedding inequalities $\|\rho_j\|_{-s} \leq \|\rho_j\|_{-s'}$ when $s = s' + 2j, j \geq 0$, and $\|\theta_j\|_{-s} \leq \|\theta_j\|_{1}.$

3 Superconvergence in a Single Space Variable

Let $\Omega = I = (0, 1), 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of $I$ with $\max(x_i - x_{i-1}) = h$, and assume that $S_h \subset C(I)$ consists of piecewise polynomial functions of degree $r$ of Lagrange type. Let $z \in (0, 1)$ be a node point of the partition, $I_0 = (0, z)$, and $I_1 = (z, 1)$.

We need to introduce some spaces that generalize $H^s(\Omega)$ in order to obtain a bound on the value of a function at the point $z$. For $s \geq 0$, define

$$\tilde{H}^s = H^{-1}(I) \bigcap H^s(I_0) \bigcap H^s(I_1); \quad |||u|||_s^2 = |||u|||_{H^s(I_0)}^2 + |||u|||_{H^s(I_1)}^2.$$ 

Obviously, $|||u|||_s = |||u|||_{s}$ when $u \in H^s(I)$. The dual norm is defined for $u \in H^1(I)$. by

$$|||u|||_{-s} = \sup \{ (u, v) / |||v|||_s; |||v|||_s \neq 0 \}$$ 

Lemma 3.1 Let $\phi \in H^1(I)$ satisfy

$$A(t; \phi, \chi) = F(\chi), \forall \chi \in S_h,$$ 

where $F: H^1(I) \rightarrow R$ is a linear functional satisfying

$$|F(\psi)| \leq M_p |||\psi|||_p, \forall \psi \in \tilde{H}^p \bigcap H^1(I), p = 1, 2, \cdots, r + 1.$$ 

Then, for $s = -1, 0, \cdots, r - 1$, we have

$$|||\phi|||_{-s} \leq C \left[ (M_1 + \inf_{\chi \in S_h} |||\phi - \chi|||_{1}) H^{s+1} + M_{s+2} \right].$$
Proof. For $g \in \tilde{H}^s(I)$, there exists a unique $\psi \in \tilde{H}^{s+2} \cap H^1_0(I)$ (see [12, Lemma 4]) such that
\[
A(t)\psi = g, \text{ in } I; \quad \|\psi\|_{s+2} \leq C\|g\|_s, \quad s \geq 0
\]
with the constant $C$ being independent of point $z$. Then, similar to the proof of Lemma 2.1, the proof can be completed by using (3.1) and
\[
\|\psi - R_h\psi\|_1 \leq C \inf_{\chi \in S_h} \|\psi - \chi\|_1 = C \inf_{\chi \in S_h} \|\psi - \chi\|_1 \\
\leq C h^{s+1}\|\psi\|_{s+2}, \quad -1 \leq s \leq r - 1.
\]

Lemma 3.2 Let $k, j \geq 0, 1 \leq q \leq r + 1$ and $-1 \leq s + 2j \leq r - 1$. Then
\[
\|\frac{\partial^k \rho_j(t,z)}{\partial t^k}\|_{-s} \leq C h^{q+r+s+2j}\left(\sum_{i=0}^{j+k} \|\frac{\partial^i u}{\partial t^i}\|_q + \sum_{i=0}^{j} \int_0^t \|\frac{\partial^i u}{\partial t^i}\|_q \, dt \right).
\]

Proof. The proof is completely similar to that of Theorem 2.1 so long as we use Lemma 3.1 and $\|\cdot\|_{-s}$ to replace Lemma 2.1 and $\|\cdot\|_{-s}$, respectively, in the argument.

Lemma 3.3 Let $k, j \geq 0, 1 \leq q \leq r + 1$ and $-1 \leq s + 2j \leq r - 1$. Then
\[
\left|\frac{\partial^k \rho_j(t,z)}{\partial t^k}\right| \leq C h^{q+r+s+2j}\left(\sum_{i=0}^{j+k} \|\frac{\partial^i u}{\partial t^i}\|_q + \sum_{i=0}^{j} \int_0^t \|\frac{\partial^i u}{\partial t^i}\|_q \, dt \right).
\]

Proof. Introduce the Green function: $G^* \in \tilde{H}^{s+2} \cap H^1_0(I)$ which satisfies
\[
A(t; G^*, \phi) = \phi(z), \quad \forall \phi \in H^1_0(I); \quad \|G^*\|_{s+2} \leq C, \quad s \geq 0,
\]
where the constant $C$ is independent of $z$. See [9] for the existence of $G^*$. Now set $\phi = \rho_j$. It follows from (2.2) (denote $\rho_{-1} = 0$) that
\[
|\rho_j(z)| \quad = \quad |A(t; \rho_j, G^*) - \int_0^t B(t, \tau; \rho_j, G^*) \, d\tau + \int_0^t B(t, \tau; \rho_j, G^*) \, d\tau|
\]
\[
= \quad |A(t; \rho_j, G^*-\chi) - \int_0^t B(t, \tau; \rho_j, G^*-\chi) \, d\tau - \left(\frac{\partial \rho_{j-1}}{\partial t}, G^*-\chi\right) + \left(\frac{\partial \rho_{j-1}}{\partial t}, G^*-\chi\right) + \int_0^t B(t, \tau; \rho_j, G^*) \, d\tau| \leq C\|\rho_j\|_1 + \int_0^t \|\rho_j\|_1 \, d\tau + \|\frac{\partial \rho_{j-1}}{\partial t}\|_{-1} \|G^*-\chi\|_1
\]
\[
+ \int_0^t \|\frac{\partial \rho_{j-1}}{\partial t}\|_{-s+2} + \int_0^t \|\rho_j\|_{-s} \, d\tau \|G^*\|_{s+2}, \quad \chi \in S_h.
\]
The proof is completed by using Theorem 2.1, Lemma 3.2 and the approximation property.

Theorem 3.1 Let $u$ and $u_h$ be solutions of the problems (1.1) and (1.3), respectively, and the initial value $u_h(0)$ be given by (2.5). Then, for $1 \leq q \leq r + 1$,
\[
|\langle u - u_h\rangle(t, z)| \leq C h^{q+r+1} \left\{ \begin{array}{ll}
\sum_{i=0}^{j} \|\frac{\partial^i u}{\partial t^i}\|_q + \sum_{i=0}^{j+1} \left(\int_0^t \|\frac{\partial^i u}{\partial t^i}\|_q^2 \, d\tau \right)^{1/2}, & 2j = r - 1, \text{ r odd,} \\
\sum_{i=0}^{j+1} \|\frac{\partial^i u}{\partial t^i}\|_q + \sum_{i=0}^{j+2} \left(\int_0^t \|\frac{\partial^i u}{\partial t^i}\|_q^2 \, d\tau \right)^{1/2}, & 2j = r - 2, \text{ r even.}
\end{array} \right.
\]
Proof. From (2.3) and $|\theta_j(t, z)| \leq \|\theta_j\|_1$, the proof is completed by using Theorem 2.2 and Lemma 3.3.

Remark 3.1 The initial condition (2.5) required in Theorems 2.3 and 3.1 can be evaluated by using no more than the data $f$ and $u_0(x)$, and the equations (1.1), (2.1) and (2.2) together with their time derivative equations.

4 Hyperbolic Integro-differential Equation

Consider the hyperbolic integro-differential equation

$$w_{tt} + A(t)w = \int_0^t B(t, \tau)w(\tau)\,d\tau + f(t, x), \quad \text{in } \Omega \times J,$$
$$w = 0, \quad \text{on } \partial \Omega \times J,$$
$$w(0) = w_0(x), w_1(0) = w_1(x), \quad \text{in } \Omega,$$

and its finite element approximation

$$\begin{cases}
(w_{ht}, \chi) + A(t; w_h, \chi) = \int_0^t B(t, \tau; w_h(\tau), \chi)\,d\tau + (f, \chi), & \chi \in S_h, \\
w_0(0), w_1(0) \in S_h.
\end{cases}$$

A quasi projection for the hyperbolic problem can be constructed in a similar manner to that in Section 2. Let the Ritz-Volterra projection $V_h$ be defined by (2.1), and $\rho_0 = V_hw - w$.

Define $\rho_j : J \rightarrow S_h$ so that, for $j \geq 1$,

$$A(t; \rho_j, \chi) = \int_0^t B(t, \tau; \rho_j(\tau), \chi)\,d\tau - (\frac{\partial^2 \rho_j}{\partial t^2}, \chi), \quad \chi \in S_h.\quad (4.3)$$

Set $\theta_0 = w_h - V_hw$, $\theta_j = w_h - V_hw - \rho_1 - \cdots - \rho_j$, $j \geq 1$. It is easy to show inductively that

$$(\frac{\partial^2 \theta_j}{\partial t^2}, \chi) + A(t; \theta_j, \chi) = \int_0^t B(t, \tau; \theta_j(\tau), \chi)\,d\tau - (\frac{\partial^2 \rho_j}{\partial t^2}, \chi), \quad j \geq 0, \quad \chi \in S_h.\quad (4.4)$$

An argument paralleling to that leading to Theorem 2.1 and Lemma 3.3 shows the following estimates: for $k, j \geq 0$, $1 \leq q \leq r + 1$ and $-1 \leq s + 2j \leq r - 1$,

$$\|\frac{\partial^k \rho_j}{\partial t^k}\|_{-s} \leq Ch^{s+q+2j}(\sum_{i=0}^{2j+k} \|\frac{\partial^i w}{\partial t^i}\|_q + \sum_{i=0}^{2j} \int_0^t \|\frac{\partial^i w}{\partial t^i}\|_q\,d\tau),\quad (4.5)$$

$$|\frac{\partial^k \rho_j(t, z)}{\partial t^k}| \leq Ch^{s+q+2j}(\sum_{i=0}^{2j+k} \|\frac{\partial^i w}{\partial t^i}\|_q + \sum_{i=0}^{2j} \int_0^t \|\frac{\partial^i w}{\partial t^i}\|_q\,d\tau).\quad (4.6)$$

Now we choose the initial values $w_h(0)$ and $w_{ht}(0)$ in the problem (4.2) such that $\theta_j(0) = \frac{\partial \theta_j}{\partial t}(0) = 0$, that is,

$$\begin{align*}
w_h(0) &= V_h w(0) + \rho_1(0) + \cdots + \rho_j(0); \\
w_{ht}(0) &= \frac{\partial V_h w}{\partial t}(0) + \frac{\partial \rho_1}{\partial t}(0) + \cdots + \frac{\partial \rho_j}{\partial t}(0).
\end{align*}\quad (4.7)$$

Lemma 4.1 Let $w_h(0)$ and $w_{ht}(0)$ be given by (4.7). Then, for $1 \leq q \leq r + 1$, we have

$$\|\theta_j\|_1 \leq \begin{cases}
Ch^{s+2j+1}(\sum_{i=0}^{2j} \|\frac{\partial^i w}{\partial t^i}\|_q^2\,d\tau)^{1/2}, & 2j \leq r - 1, \\
Ch^{s+2j+2}(\sum_{i=0}^{2j+2} \|\frac{\partial^i w}{\partial t^i}\|_q^2 + \sum_{i=0}^{2j+3} \int_0^t \|\frac{\partial^i w}{\partial t^i}\|_q^2\,d\tau)^{1/2}, & 2j \leq r - 2.
\end{cases}\quad (4.8)$$
Proof. Taking $\chi = \frac{\partial \theta_j}{\partial t}$ in (4.4) and integrating from 0 to $t$, we obtain
\[
\frac{1}{2} \|rac{\partial \theta_j}{\partial t}\|^2 + \frac{1}{2} A(t; \theta_j, \theta_j) - \frac{1}{2} \int_0^t A_t(t; \theta_j(\tau), \theta_j(\tau)) \, d\tau
= \int_0^t B(t, \tau; \theta_j(\tau), \theta_j(t)) \, d\tau - \int_0^t B(t, \tau; \theta_j(\tau), \theta_j(t)) \, d\tau
- \int_0^t \int_0^\tau B_t(\tau, s; \theta_j(s), \theta_j(\tau)) \, ds \, d\tau - \int_0^t \left( \frac{\partial^2 \rho_j}{\partial t^2}, \frac{\partial \theta_j}{\partial t} \right) \, d\tau.
\]
It follows from Cauchy inequality that
\[
\left\| \frac{\partial \theta_j}{\partial t} \right\|^2 + \left\| \theta_j(t) \right\|^2 \leq C \int_0^t \left\| \theta_j(\tau) \right\|^2 \, d\tau + C \int_0^t \left\| \frac{\partial^2 \rho_j}{\partial t^2} \right\|^2 \, dt.
\]
Applying Gronwall Lemma and (4.5) with $s = 0$, $k = 2$, we obtain the first inequality in Lemma 4.1. If we integrate the last right hand term in (4.8) by parts in time to obtain
\[
\int_0^t \left( \frac{\partial^2 \rho_j}{\partial t^2}, \frac{\partial \theta_j}{\partial t} \right) \, dt = \left( \frac{\partial^2 \rho_j}{\partial t^2}, \theta_j \right) - \int_0^t \left( \frac{\partial \rho_j}{\partial t^2}, \theta_j \right) \, dt 
\leq \left( \left\| \frac{\partial^2 \rho_j}{\partial t^2} \right\|_{-1} + \int_0^t \left\| \frac{\partial^{3} \rho_j}{\partial t^3} \right\|_{-1} \, dt \right) \left\| \theta_j \right\|_1,
\]
then, the same argument produces the second inequality in Lemma 4.1. The proof is completed.

**Theorem 4.1** Let $w(t)$ and $w_h(t)$ be solutions of the problems (4.1) and (4.2), respectively, and the initial values $w_h(0)$ and $w_{h,t}(0)$ be given by (4.7). Then, for $1 \leq q \leq r + 1$, we have
\[
\left\| w - w_h \right\|_{-s} \leq C h^{q+s} \left\{ \begin{array}{l}
\sum_{i=0}^{2j} \left\| \frac{\partial^i w}{\partial t^i} \right\|_q + \sum_{i=0}^{2j+2} \left( \int_0^t \left\| \frac{\partial^i w}{\partial t^i} \right\|_q^2 \, d\tau \right) \frac{1}{2}, \quad 0 \leq s \leq 2j \leq r - 1, \\
\sum_{i=0}^{2j+2} \left\| \frac{\partial^i w}{\partial t^i} \right\|_q + \sum_{i=0}^{2j+3} \left( \int_0^t \left\| \frac{\partial^i w}{\partial t^i} \right\|_q^2 \, d\tau \right) \frac{1}{2}, \quad 0 \leq s \leq 2j + 1 \leq r - 1.
\end{array} \right.
\]
Moreover, in the case of a single variable, we have
\[
\left| (w - w_h)(t, z) \right| \leq C h^{q+r-1} \left\{ \begin{array}{l}
\sum_{i=0}^{2j} \left\| \frac{\partial^i w}{\partial t^i} \right\|_q + \sum_{i=0}^{2j+2} \left( \int_0^t \left\| \frac{\partial^i w}{\partial t^i} \right\|_q^2 \, d\tau \right) \frac{1}{2}, \quad 2j = r - 1, \quad r \text{ odd,}
\\
\sum_{i=0}^{2j+2} \left\| \frac{\partial^i w}{\partial t^i} \right\|_q + \sum_{i=0}^{2j+3} \left( \int_0^t \left\| \frac{\partial^i w}{\partial t^i} \right\|_q^2 \, d\tau \right) \frac{1}{2}, \quad 2j = r - 2, \quad r \text{ even.}
\end{array} \right.
\]

**Proof.** Note that $w_h - w = w_h - V_h w + V_h w - w = \theta_j + \rho_0 + \cdots + \rho_j$. Then the proof can be completed by using (4.5) - (4.6) and Lemma 4.1.

**References**


