A Novel 4-DOF Parallel Manipulator
And Its Kinematic Modelling

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Abstract
In this paper, a novel four-degree-of-freedom (4-DOF) hybrid parallel platform manipulator is presented. The movable platform of the parallel manipulator can translate along two directions and rotate around two axes respectively. The kinematics model is formulated, which describes the inverse kinematic transformation and forward kinematic transformation. A 5-axis parallel machine tool based on the 4-DOF parallel manipulator is presented in this paper.

1. Introduction

From the first ideas of parallel mechanisms proposed by Gough[1] and Stewart[2], parallel mechanisms —to this family belongs the Stewart platform — have been found in many industrial applications, such as the octahedral hexapod machine tool developed by Ingersoll Milling Machines[3], the M-850 Hexapod six-axis micro-positioning system (DC motor driven) by Physik Instrument (PI)[4], multiple-DOF active vibration control systems[5], flight/motion simulators[1] and many other applications[6-7]. In the last 20 years, a lot of interesting mechanical devices and design methods have been extensively studied[8-13]. However, most efforts are dedicated to 6-DOF mechanisms which are now well known or 3-DOF mechanisms[14].

There is a need for equipment providing more than 3 DOF’s arranged in parallel and based on simpler arrangements than 6-DOF arrangements in application. As a matter of fact, the cutting process requires only 5 controlled axes —plus the spindle rotation— in order that a free surface in the workspace of a machine tool can be machined. One can provide the spindle with 4 axes by a 4-DOF parallel manipulator and let the workbench on which work-piece is fixed be moved by an additional 5th axis. In this way, the machine tool controlled in simultaneous 5 axes control mode can be made easily.

This paper presents a novel 4-DOF hybrid parallel platform manipulator. The movable platform of the manipulator can translate along two directions and rotate around two axes respectively. A 5-axis parallel machine tool has been developed based on the 4-DOF parallel manipulator presented in this paper.

Follow the introduction, section 2 proposes the structure of the new parallel platform. In section 3 and section 4, inverse and forward kinematics of the new parallel manipulator are discussed respectively. Section 5 introduces the development of prototype. At last, section 6 is the conclusion.

2. Mechanism description

The schematic view of the 4-DOF parallel manipulator is shown in Figure 1. The movable platform of the mechanism is made up of two parts—part a and part b. The part a and part b are connected with a rotate pair
whose axis is along \( p - p \). The platform is suspended by four rods that are attached to the platform with joints at \( t_1, t_2, t_3 \) and \( t_4 \) respectively. The other ends of the four rods are similarly attached to four sliders with joints at \( s_1, s_2, s_3 \) and \( s_4 \) respectively. The four sliders can translate up and down according to the fixed guideways. Where the motion pairs at \( s_1, s_2, s_3 \) and \( s_4 \) are 1-DOF rotate pairs (R). The motion pairs at \( s_1, s_2 \) are 2-DOF universal pairs (U). The motion pairs at \( t_1, t_2 \) are 3-DOF spherical pairs (S).

![Diagram of the 4-DOF parallel manipulator](image)

**Fig. 1** The 4-DOF parallel manipulator

According to Kutzbach Crubler’s formula, the mobility (i.e. DOF) of the mechanism can be computed as follows:

\[
M = d(n - g - 1) + \sum_{i=1}^{g} f_i
\]

(1)

Where \( M \) stands for the mobility of the mechanism, \( d \) for the order of the mechanism, \( n \) for the number of rods, \( g \) for the number of motion pairs, \( f_i \) for the DOF of the \( i \)th motion pair.

The loop \( s_3 t_3 s_4 \) which includes part \( a \) has 3 common constraints and its order is 3. Therefore, the DOF of part \( a \) is:

\[
M_a = 3(6 - 6 - 1) + (1 + 1 + 1 + 1 + 1) = 3
\]

The loop \( s_1 t_3 s_2 \) which includes part \( b \) is free of common constraint and its order is 6. Therefore, the DOF of part \( b \) is:

\[
M_b = 6(6 - 6 - 1) + (1 + 1 + 2 + 2 + 3 + 3) = 6
\]

The part \( a \) and part \( b \) are connected by a rotate pair (whose number of constraints is 5). The mobility of the mechanism is:

\[
M = M_a + M_b - 5 = 4
\]

Therefore the motion of the mechanism is certain given certain inputs at slider \( s_1, s_2, s_3 \) and slider \( s_4 \) respectively.

Because of the special mechanism design it is easy to recognize that the parallel structure is of 3 DOF’s: The \( s_3 t_3 s_4 \) composes a single-loop planar six-bar linkage on plane \( OYZ \). The part \( a \) in the six-bar linkage has 3 DOF’s, which are described by the two translations along \( X \) and \( Z \) directions and by the rotation about the axis parallel to \( X \)-direction. The part \( b \) can rotate about \( p - p \) axis with respect to the part \( a \).

Therefore, the 4 DOF’s of the part \( b \) are two translations and two rotations.

![Diagram of the link-pair relationship](image)

**Fig. 2** The link-pair relationship diagram of the parallel platform

The link-pair relationship diagram of the parallel platform is shown in Figure 2. Where the letter R is equivalent to a rotational joint, P to a prismatic joint, S to a spherical joint and U to a universal joint. Letter P
3. Inverse kinematics

The inverse kinematics can be described as: given the position and orientation of the movable platform, calculate the displacements of the actuating devices which can be used to attain this given position and orientation\textsuperscript{151}. First it will be necessary to establish a set of coordinate frames. As with conventional robots, a base frame and two movable frames will be defined. With reference to Figure 1, the movable frame 1 \((O'-X'Y'Z')\) lies on the part \(b\) of the movable platform and the orient \(O'\) is at the center of the axis around which part \(b\) rotates with respect to part \(a\). Also, \(O'\) is on the rotating axis of the tool which is mounted on the movable platform (part \(b\)). The movable frame 2 lies on the part \(a\) of the movable platform which is superposed with movable frame 1 when the movable frame (both part \(a\) and part \(b\)) is on the horizontal plane (The movable frame 2 is not shown in Figure 1). The base frame \((O-XYZ)\) lies on the base platform whose orientation is same as the orientation of the movable frame 1 \((O'-X'Y'Z')\) and the orient \(O\) located the axis \(Z\) of the movable frame 1 \((O'-X'Y'Z')\) when the movable platform (part \(a\) and part \(b\)) is on the horizontal plane.

It is evident that the part \(b\) of the movable platform can translate along \(Y\)-direction or \(Z\)-direction with respect to the base frame and rotate around \(X'\)-axis or \(Y'\)-axis with respect to the movable frame.

Let \(R\) and \(P\) denote the orientation and position of the movable frame 1 \((O'-X'Y'Z')\) with respect to the base coordinate frame \((O-XYZ)\). \(R\) is a \(3 \times 3\) matrix and \(P\) is a \(3 \times 1\) vector. Because of the structure of this manipulator, \(P\) and \(R\) can be expressed as:

\[
P = (p_x, p_y, p_z)^T = (0, p_y, p_z)^T
\]

\[
R = R_x R_y.
\]

Where \(R_x\) is the rotation matrix which denotes that the movable platform rotates angle \(A\) around \(X'\)-axis. \(R_y\) is the rotation matrix which denotes that the movable platform rotates angle \(B\) around \(Y'\)-axis.

\[
R_x = \begin{bmatrix}
1 & 0 & 0 \\
0 & cA & -sA \\
0 & sA & cA
\end{bmatrix}
\]

\[
R_y = \begin{bmatrix}
cB & 0 & sB \\
0 & 1 & 0 \\
-sB & 0 & cB
\end{bmatrix}
\]

Therefore:

\[
R = R_x R_y = \begin{bmatrix}
cB & 0 & sB \\
-sAcB & cA & -sACB \\
-sBCA & sA & cACB
\end{bmatrix}
\]

Where \(cA\) denotes \(\cos(A)\), \(sA\) denotes \(\sin(A)\), etc.. It is evident that the orientation and position of the movable frame 2 with respect to the base coordinate frame \((O-XYZ)\) are \(P\) and \(R_x\) respectively.

Now, the four joints of the movable platform are denoted as \(t_1, t_2, t_3, t_4\). The coordinates of \(t_i\) \((i=1, 2, 3, 4)\) in each own movable frame are described by the vectors:

\[
t_i' = (x_i, y_i, z_i)^T \quad i=1\sim4
\]

Then, the coordinates of \(t_i\) \((i=1, 2, 3, 4)\) in base frame can be described as:

\[
t_i = R t_i' + P \quad i=1,2 \quad (4)
\]

\[
t_i = R_x t_i' + P \quad i=3,4 \quad (5)
\]

Or

\[
\begin{bmatrix}
t_{1x} \\
t_{1y} \\
t_{1z}
\end{bmatrix} = \begin{bmatrix}
x_1 \\
y_1 + p_y \\
z_1 + p_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
t_{2x} \\
t_{2y} \\
t_{2z}
\end{bmatrix} = \begin{bmatrix}
x_2 \\
y_2 + p_y \\
z_2 + p_z
\end{bmatrix}
\]

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\[
\begin{bmatrix}
   t_{3y} \\
   t_{3z} \\
   t_{4y} \\
   t_{4z}
\end{bmatrix} = R_X \begin{bmatrix}
   x_3 \\
   y_3 \\
   z_3 \\
   p_z
\end{bmatrix} + \begin{bmatrix}
   0 \\
   0 \\
   0 \\
   p_z
\end{bmatrix}
\]

Let \( L_1, L_2, L_3, L_4 \) denote the length of the four rods respectively, the following constraint equations represent the inverse kinematics of the platform manipulator:

\[
L_i^2 = \| s_i - t_i \|^2 \quad i=1-4
\]

Or

\[
L_i^2 = (s_{ix} - t_{ix})^2 + (s_{iy} - t_{iy})^2 + (s_{iz} - t_{iz})^2
\]

Where

\[
s_{ix} = H_i
\]

Therefore, the displacement of each slider is as follows:

\[
\begin{align*}
H_1 &= t_{ix} + \sqrt{L_i^2 - (s_{ix} - t_{ix})^2 - (s_{iy} - t_{iy})^2} \\
H_2 &= t_{iz} + \sqrt{L_i^2 - (s_{ix} - t_{ix})^2 - (s_{iy} - t_{iy})^2} \\
H_3 &= t_{iy} + \sqrt{L_i^2 - (s_{ix} - t_{ix})^2 - (s_{iy} - t_{iy})^2} \\
H_4 &= t_{iz} + \sqrt{L_i^2 - (s_{ix} - t_{ix})^2 - (s_{iy} - t_{iy})^2}
\end{align*}
\]

### 4. Forward kinematics

The forward kinematics problem of the 4-DOF parallel manipulator can be stated as follows: given the displacements \( H_i \) (i=1-4) of the actuating devices, compute the position \((0, p_x, p_y)\) and orientation \((A, B)\) of the movable platform with respect to the fixed-base reference frame.

By putting the coordinates of \( t_i \) into equation (6), yields four equations with four variables \( p_x, p_y, A, B \),

\[
f_i(p_x, p_y, A, B) = 0 \quad i=1-4
\]

Where

\[
\begin{align*}
f_1 &= -L_1^2 + s_{1x}^2 + s_{1y}^2 + H_1^2 + x_1^2 + y_1^2 + z_1^2 \\
&- 2s_{1x}p_x - 2H_1p_y + p_y^2 + p_z^2 - 2s_{1y}x_1cB \\
&- 2s_{1y}y_1cA - 2s_1y_1sAsB + 2s_{1x}z_1AcB + 2H_1x_1cAsB - 2H_1z_1cAcB + 2y_1p_xcA + 2p_xycAsA + 2x_1p_ycA \\
&- 2p_xz_1cAcB - 2p_xx_1cAsB + 2p_zz_1cAcB & = 0
\end{align*}
\]

\[
\begin{align*}
f_2 &= -L_2^2 + s_{2x}^2 + s_{2y}^2 + H_2^2 + x_2^2 + y_2^2 + z_2^2 \\
&- 2s_{2x}p_x - 2H_2p_y + p_y^2 + p_z^2 - 2s_{2y}x_2cB \\
&- 2s_{2y}y_2cA - 2s_2y_2sAsB + 2s_{2x}z_2AcB + 2H_2x_2cAsB - 2H_2z_2cAcB + 2y_2p_xcA + 2p_xycAsA + 2x_2p_ycA \\
&- 2p_xz_2cAcB - 2p_xx_2cAsB + 2p_zz_2cAcB & = 0
\end{align*}
\]

\[
\begin{align*}
f_3 &= -L_3^2 + s_{3x}^2 + s_{3y}^2 + H_3^2 + x_3^2 + y_3^2 + z_3^2 \\
&- 2s_{3x}x_3 - 2s_{3y}y_3 - 2H_3p_y + p_y^2 + p_z^2 \\
&- 2s_{3y}y_3cA + 2s_{3z}z_3sA - 2H_3y_3sA - 2H_3z_3cA + 2y_3p_xcA - 2z_3p_ycA + 2y_3pcA & = 0
\end{align*}
\]

\[
\begin{align*}
f_4 &= -L_4^2 + s_{4x}^2 + s_{4y}^2 + H_4^2 + x_4^2 + y_4^2 + z_4^2 \\
&- 2s_{4x}x_4 - 2s_{4y}y_4 - 2H_4p_y + p_y^2 + p_z^2 \\
&- 2s_{4y}y_4cA + 2s_{4z}z_4sA - 2H_4y_4sA - 2H_4z_4cA + 2y_4p_xcA - 2z_4p_ycA + 2y_4pcA & = 0
\end{align*}
\]

It is noted that the locations for the link attachment points \( t_i \) (i=1-4) can be assigned by design. Careful choice for the locations can lead to simpler kinematic transformation. We choose \( t_i \) (i=1-4) such that the origin \( O^\prime \) of the frame \((O^\prime - X^\prime Y^\prime Z^\prime)\) and the \( t_i \) (i=1-4) lie on a plane, i.e.,

\[
t'_i = (x_i, y_i, z_i)^T = (x_i, y_i, 0)^T \quad i=1-4
\]

In that case, the \( f_i \) (i=1-4) reduces to
\[ f_1 = -L_1^2 + s_{1x}^2 + s_{1y}^2 + H_1^2 + x_1^2 + y_1^2 - 2s_{1y}p_y \\
-2H_1p_z + p_y^2 + p_z^2 - 2s_{1z}x_1cB - 2s_{1y}y_1cA \\
-2H_1y_1sA - 2s_{1y}x_1sAsB + 2H_1x_1cAsB + 2y_1p_ycA \\
+ 2p_zy_1sA + 2x_1p_ycAsB - 2p_zx_1cAsB = 0 \]

\[ f_2 = -L_2^2 + s_{2x}^2 + s_{2y}^2 + H_2^2 + x_2^2 + y_2^2 - 2s_{2y}p_y \\
-2H_2p_z + p_y^2 + p_z^2 - 2s_{2z}x_2cB - 2s_{2y}y_2cA \\
-2H_2y_2sA - 2s_{2y}x_2sAsB + 2H_2x_2cAsB + 2y_2p_ycA \\
+ 2p_zy_2sA + 2x_2p_ycAsB - 2p_zy_2cAsB = 0 \]

\[ f_3 = -L_3^2 + s_{3x}^2 + s_{3y}^2 + H_3^2 + x_3^2 + y_3^2 - 2s_{3y}x_3 \\
-2s_{3y}p_y - 2H_3p_z + p_y^2 + p_z^2 - 2s_{3y}y_3cA \\
-2H_3y_3sA + 2y_3p_ycA + 2y_3p_zcA = 0 \]

\[ f_4 = -L_4^2 + s_{4x}^2 + s_{4y}^2 + H_4^2 + x_4^2 + y_4^2 - 2s_{4z}x_4 \\
-2s_{4x}p_y - 2H_4p_z + p_y^2 + p_z^2 - 2s_{4z}y_4cA \\
-2H_4y_4sA + 2y_4p_ycA + 2y_4p_zcA = 0 \]

It is noted that the choice of (9) can be made without loss of generality. Therefore, equations (8) represent the forward kinematics of the platform manipulator. They are four nonlinear high-order equations with four variables which can be solved by Newton-Raphson algorithm \(^{[16]}\).

5. Prototype development

A 5-axis parallel machine tool has been developed based on the 4-DOF parallel manipulator presented in this paper. The schematic view of the machine tool is shown in Figure 3. The five axis motions of the machine tool are \(X, Y, Z, A, B\) respectively. Basically the machine tool consists of a workbench, a bed of machine tool, and a 4-DOF parallel manipulator. The main motion of the machine tool is the rotation of the tool that is driven by the motor spindle 3 mounted on the movable platform of the parallel manipulator. Among the five feed motions, the workbench on which work-piece is fixed is responsible for the translation along \(X\)-direction and the other feed motions \((Y, Z, A, B)\) are realized by the movable platform of the parallel manipulator.

![Fig. 3 The schematic view of the machine tool](image)

In order to decrease the axial resistance of the rotate pairs at \(s_1, t_1, t_4, s_4\) and enhance the rigidity of the machine tool, an additional planar six-bar linkage 7 is used to constraint part \(a\) of the movable platform to move on a plane. The connection relationship of the additional six-bar linkage and the other links of the machine tool is shown in Figure 2 (dashed line). The top link of the six-bar linkage is also the part \(a\) of the movable platform.

Compared with the other parallel mechanism based machine tool, the present machine tool has some characteristics as follows:

1. The special mechanism arrangements enlarged the workspace of the machine tool greatly in \(X\)-direction and \(Z\)-direction;
2. It is convenient to handle the work-piece;
3. The description of the workspace is similar to that of the conventional machine tool.
6. Conclusion

A novel 4 degree-of-freedom (DOF) hybrid parallel platform manipulator is presented. The movable platform of the parallel manipulator can translate along two directions and rotate around two axes respectively. The kinematics model is formulated, which describes the inverse kinematics and forward kinematics. A 5-axis parallel machine tool based on the 4-DOF parallel manipulator is presented in this paper. The machine tool is now in operation at Shenyang Institute of Automation, Chinese Academy of Sciences, in order to evaluate its performance further.

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