Generalized Point Wise Min-Norm Control Based on Control Lyapunov Functions

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Abstract: A new nonlinear controller design method based on Control Lyapunov Functions, called generalized point wise min-norm controller, is presented in this paper, which is a generalized version of Freeman’s point wise min-norm controller in 1996. And the continuity of the new controller is proved using the set valued analysis theory. The greatest improvement of the new controller, compared with the Freeman’s controller, is its greatly improved designing flexibility from the induced guide function. And it is shown that the new control method can be used together with some other controller design method with special performance index by the concept of guide function, especially those that the closed loop stability cannot be ensured sufficiently. Finally, an example is given combining with a linearized controller to both enlarge the large scale stability and preserve the local performance of that.

Key Words: Nonlinear System, Control Lyapunov Function, Linearization

1 INTRODUCTION

Lyapunov analysis theory has been playing a major role in the stability analysis of nonlinear system for a long time. Given a general nonlinear system, if a positive definite function \( V(x) \) can be found such that the derivative of \( V(x) \) along the trajectory of the system is negative, the system can be guaranteed to be asymptotically stable [1]. However, the application of Lyapunov theory in nonlinear controller design is difficult and limited. In most situations, a controller is first given, and then the closed loop stability is validated by searching a Lyapunov function of it. We call this an indirect method to use Lyapunov analysis in nonlinear controller design.

On the other hand, the direct method to use Lyapunov theory in the synthesis problem of the nonlinear controller is Control Lyapunov Function (CLF) based method. A CLF of a nonlinear system is a positive definite function of states that can be made negative in each state, by some feasible inputs. Arstren first researched the problem of existence of CLF and the stabilizability of a nonlinear system [2]. However, he did not give any method to design such a controller. Later in 1989, Sontag showed that if a CLF is known for a nonlinear system that is affine in the control, then the CLF and the system equations can be used to find formulas that render the system asymptotically stable [3]. And in 1996, Freeman and Kokotovic reproved the results of Arstren by using set valued analysis, and introduced another famous formula, called point wise min-norm (PMN) formula [4-5].

It has been shown that Sontag’s formula and the min-norm formula are good universal formulas in the sense that they enjoy certain stability margins and are inverse optimal. However, one of the drawbacks with the formula of Sontag and Freeman is that they can not provide parameters convenient enough to regulate the performance of the controller. Actually, Sontag’s formula does not provide any convenient parameters to tune the performance of the controller. The only available tuning device is to modify the CLF itself. Unfortunately, how to obtain a CLF is itself a difficult problem, not to mention a proper CLF. Although both the modified version of Sontag’s formula by Freeman and the PMN formula [5] have a positive definite function to regulate the performance of the closed loop, the deficiency of proper regulating strategies make it difficult to implement.

In this paper, we generalized the Freeman’s PMN controller [5] by introducing a guide function into it to obtain a new nonlinear controller design method based on CLF, called Generalized Point wise Min-Norm controller (GPMN). The predefined CLF is used to ensure the closed loop stability, simultaneously, the guide function is used to guide the selection of the control input in each state in order to make the closed loop have some required performance.

2 THE CONCEPT OF CLF AND SOME CORRESPONDING RESULTS

The nonlinear system under consideration is

\[
\dot{x} = f(x) + g(x)u \quad u \in U \subset \mathbb{R}^n \tag{1}
\]

where \( x \) are states; the vector field \( f \) and \( g \) are both smooth function, \( f(0) = 0 \); \( U \) is the convex control constraint.

We have mentioned that CLF is a successful attempt of using Lyapunov analysis method to the controller design of nonlinear systems. A simple explanation of a CLF of system (1) is that it is a positive definite function \( V(x) \) of states such that in every state point, at least one control input can be found to make the derivative of \( V(x) \) be negative, along with the trajectory of system (1). And the detailed definition of CLF can be denoted as fol-
lowing.

**Definition I-1**

If there exists a $C_1$ function $V(x)$: $\chi \subset \mathbb{R}^n \to \mathbb{R} + \cup \{0\}$, such that

- $V(0) = 0$, $V(x) > 0$ if $x \neq 0$
- $\alpha_1(||x||) < V(x) < \alpha_2(||x||)$
- $\inf_{n \in \mathbb{N}} V’(x) f(x) + V’(x) g(x) u < 0$, $\forall x \in \chi - \{0\}$

where $\alpha_1()$ and $\alpha_2()$ are class $K_\infty$ function.

Then we call $V(x)$ a CLF of system (1). If furthermore $\chi = \mathbb{R}^n$, $V(x)$ is called a global CLF of system (1).

From Definition I-1, it is a fact that: if we can obtain a CLF of a nonlinear system, then in every ‘feasible’ state point, a “permitted” control action set can be found, and thus, what we should do next to obtain a stable controller is only to find a strategy to select a series of control actions in the series ‘permitted’ control sets, see Fig.1.

![Fig. 1 sketch of CLF](image)

*The shadow in this figure indicates the ‘permitted’ set of $(x, u)$ in which the derivative of $V(x)$ with respect to time, along with system (1), is negative. And the dashed line means the states where $V(x)g(x) = 0$. If the selected control law can be ensured to be always in this shadow, the closed loop is stable.

As we all known, continuous feedback controller is necessary in real application, and thus, researchers are trying their best to find continuous controller based on a known CLF. In 1983, Astrein first published his result about the equivalence of continuous stabilizability of a nonlinear plant and the existence of a CLF of it. However, he did not give out a method to find such a stable controller. And Sontag and Freeman solve this problem in 1989 and 1996 respectively.

Sontag’s idea is originated from the calculating formulas of roots for equation of 2-th order, and can be written as following through slightly variation by Freeman [5]:

$$ u = \begin{cases} \frac{V_f g + \sqrt{(V_f g)^2 + q(x)(V_g g^T V_f^T)}}{V_g g^T V_f^T}, & V_g \neq 0 \\ 0, & V_g = 0 \end{cases} $$

where $q(x)$ is a positively definite function and continuous everywhere except possibly at $x = 0$.

The Freeman’s PMN formula controller, is as

$$ \min_{u} ||u|| $$

s.t. $V_f[f(x) + g(x) u] \leq -\sigma(x)$

where $\sigma(x)$ is a positively definite and continuous function. The controller (3) can also be explicitly denoted as

the following equation from projection theorem if the input constraint set $U$ is selected large enough,

$$ u = \begin{cases} \frac{[V_f f + \sigma(x)] V_f^T}{V_g g^T V_f^T}, & V_f f + \sigma(x) > 0 \\ 0, & V_f f + \sigma(x) \leq 0 \end{cases} $$

(4)

One of the drawbacks of the preceding two formulas is that neither of them provides enough parameters and corresponding regulatory rules. Thus, in the following sections, we will give a new controller design formulation to parameterize Freeman’s PMN controller, as well, to make it convenient to be used in real application.

3 GENERALIZED POIN WISE MIN–NORM CONTROL

3.1 The Form of Controller

In this section, we will give the generalized version of Freeman’s point wise min-norm controller by introducing a guide function. The main result can be denoted as the following theorem.

**Theorem I** If $V(x)$ is a CLF of system (1) in the set of $\Omega$:

$$ \Omega = \{x \in \mathbb{R}^n | V(x) \leq c\} $$

$\zeta(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function of states such that $\zeta(0) = 0$; and $\sigma(x)$ is a positive definite and continuous function. Then the following controller,

$$ u_{i(x)}(x) = \arg \min_{n \in K_i(x)} ||u - \zeta(x)|| $$

(5)

$K_i(x) = \{y | V_f(x) f(x) + V_g(x) g(x) y < \sigma(x), y \in U\}$

called GPMN controller, is continuous in $\Omega$, except possibly at $x = 0$, and can stabilize system (1). Furthermore, if $V(x)$ satisfies the following continuous control property,

For every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $||x|| < \delta$ the inequality $V_f(x) f(x) + V_g(x) g(x) u < 0$ holds for a certain $u$ with $||u|| < \varepsilon$.

Then, controller (5) can be ensure continuous in $\Omega$.

**Proof of Theorem I** Let $V(x)$ be a Lyapunov function candidate, and the derivative of $V(x)$ is

$$ \dot{V}(x) = V_f f(x) + V_g g(x) u $$

From Eq.(6), it is known that $u \in K_i(x)$, and we have

$$ \dot{V}(x) = V_f f(x) + V_g g(x) u_{i(x)} \leq \sigma(x) $$

Thus, the positive definitions of $\sigma(x)$ ensure the derivative of $V(x)$ is negatively definite, so controller (5) can stabilize system (1).

The proof of the continuity of controller (5) need to use the concept of set-valued analysis theory, and the detailed proof can be seen in next section.

**End of Proof**

If there are no input constraints, (or the input constraint set is big enough), we can obtain an analytic expression of (5) as the following form by projection theory

$$ 405 $$
\[ u(x) = \begin{cases} 
    [V_f + \sigma + V_g \xi(x)]g^TV_x^T + \xi(x), & \text{if } V_f + \sigma + V_g \xi(x) > 0 \\
    \xi(x), & \text{if } V_f + \sigma + V_g \xi(x) \leq 0 
\end{cases} \]

\textbf{Remark} The sketch of the PMN controller and GPMN controller (5) or (6) can be denoted as Fig.2, from which we can see that, for PMN controller (the left figure of Fig.2), the control input of every state has the minimizing ‘permitted’ distant from zero, whereas for GPMN controller (the right figure of Fig.2), the control input of every state has nearest distance from the guide function \( \xi(x) \).

**Definition III-2** Let \( T \) is a set valued map from \( X \) to \( Y \), then, we shall say that \( T \) is closed – valued (convex – valued) map if the images of \( T \) are closed (convex).

**Definition III-3** A set valued map \( T: X \rightarrow Y \) is called upper semicontinuous (usc) at \( x \in \text{Dom}(T) \) if and only if for any neighborhood \( U \) of \( T(x) \),

\[ \exists \eta > 0 \text{ such that } \forall x' \in B(x, \eta), \ T(x') \subset U \]

It is said to be usc if and only if it is usc at any point of \( \text{Dom}(T) \). \( T \) is called lower semicontinuous (lsc) if \( \forall x \in \text{Dom}(T) \) and only if for every sequence \( \{x_i\} \in X \) converging to \( x \) and every \( y \in T(x) \) there exists a sequence \( \{y_i\} \in Y \) converging to \( y \) and \( N \geq 1 \) such that \( y_i \in T(x_i) \) for all \( i \geq N \). It is said to be lsc if and only if it is lsc at any point of \( \text{Dom}(T) \).

Continuous selection of a set valued map is an important content in set valued analysis. And a famous continuous selection of a set valued map is named minimal selection

\[ m(T(x)) := \{ u \in T(x) \mid |u| = \min_{y \in T(x)} |y| \} \]

The continuity of the minimal selection \( m(T(x)) \) can be denoted as following Lemma.

**Lemma III-1** Let \( T \) be a set valued map from a metric space \( X \) to a Banach space \( Y \) that is strictly convex and reflexive. And \( T \) is a closed – valued and convex – valued map such that

i) \( T \) is lsc;

ii) The graph of \( T \) is closed;

iii) \( m(T, x) \) is contained in a compact subset of \( Y \) then the minimal selection is continuous.

The preceding definition and lemma are all from [6].

First, we give the following Theorem.

**Theorem III-1** Let \( T \) be a set valued map from a metric space \( X \) to a Banach space \( Y \) that is strictly convex and reflexive. And \( T \) is a closed – valued and convex – valued map. If \( \zeta(x): X \rightarrow Y \) is a single function. Then the newly defined set valued map, \( R: X \rightarrow Y \) \( (R(x) = T(x) - \zeta(x)) \) such that

i) \( R \) is closed – valued and convex – valued map;

ii) \( \zeta(x) \) is continuous function, and \( T \) is lsc, so does \( R \);

iii) If the graph of \( T \) is closed, so does the graph of \( R \).

**Proof of Theorem III-1**: The proof of i can be directly completed by the definition of III-2.

If \( T \) is lsc, from definition III-3, for every sequence \( \{x_i\} \in X \) converging to \( x \) and every \( y \in X \) there exists a sequence \( \{y_i\} \in Y \) converging to \( y + \zeta(x) \) and \( N \geq 1 \) such that \( y_i \in T(x_i) \) for all \( i \geq N \). On the one hand, from the definition of \( R(x, y) = y - \zeta(x) \) and \( R(x) = T(x) \), and from the continuity of \( \zeta(x) \), \( \zeta(x) \) converges to \( \zeta(x) \), thus, we conclude that the sequence \( \{y_i - \zeta(x)\} \in Y \) converges to \( y \), and when \( i \geq N \), \( y_i \in R(x) \). Thus, from definition III-3, we have proved ii.

Now we will prove that if the graph of \( T \) is closed, so does \( R \). For every \( (t, l) \in X \times Y \) Graph(\( R \)), we have \( (t, l - \zeta(t)) \in X \times Y \) Graph(\( T \)), from the closing of graph of \( T \), \( X \times Y \) Graph(\( T \)) is open, thus there exists a positive number \( \varepsilon \) such that \( B((t, l - \zeta(t)), \varepsilon) \subset X \times Y \) Graph(\( T \)). From the definition of \( R \), \( B((t, l), \varepsilon) \subset X \times Y \) Graph(\( R \)).

406
thus, \(X \times Y\)–Graph(\(R\)) is open in the product space \(X \times Y\), i.e., Graph(\(R\)) is closed.

**End of Proof**

**The Proof of the second part of Theorem 1:** By the existence of the CLF, there exists \([4]\) a positive definite function \(\sigma(x)\) such that

\[
L_{\sigma}(x) = \{ u \in \mathbb{R}^n : V_f(x) + V_g(x)u \leq -\sigma(x) \} \\
K_{\sigma}(x) = L_{\sigma}(x) \cap U
\]

is closed – valued and convex – valued lsc map, and the graph of \(K_{\sigma}\) is closed in the following state set

\[\Omega = \text{int}(\Omega)\]

where \(c\) is an arbitrary positive number smaller than \(c\). And int(\(\Omega\)) means the set composed by all the interior points of \(\Omega\).

From Theorem III-1, the set valued map \(K_{\sigma}(x) – \zeta(x)\) is convex-valued and convex – valued lsc map, and the graph of \(K_{\sigma}(x) – \zeta(x)\) is closed in \(\Omega, \text{int}(\Omega)\). And by simple derivation we can conclude that controller (5) is the minimal selection of set valued map \(K_{\sigma}(x) – \zeta(x)\), which will be denoted as \(m(x)\) in the following. Thus, by lemma 9-3-1 of \([6]\), \(x \rightarrow |m(x)|\) is usc, which implies that \(m(x)\) is locally bounded. For every \(x' \in \Omega, \text{int}(\Omega)\), and every positive number \(\delta\),

\[
\|m(x')\|_0 \leq \|K_{\sigma}(x) – \zeta(x)\| = \|m(x)\|_0 + \|\zeta(x)\|_0 = \|m(x)\|_0 + \|\zeta(x)\|_0
\]

is bounded. Lemma III-1 implies that \(m(x)\) is continuous in \(x'\), the element \(x' \in \Omega, \text{int}(\Omega)\) being arbitrary, the continuity of \(m(x)\) in \(\Omega, \text{int}(\Omega)\) is proved.

Next suppose \(V\) satisfies the ccp; by the conclusion of Arstien in \([2]\), there exists a continuous controller \(k(x)\) with \(k(0) = 0\). From the definition of \(m(x)\), the continuity of \(\zeta(x)\), and \(\zeta(0) = 0\), we have \(0 \leq |m(x)| \leq |\zeta(x)|\) for all \(x \in \Omega, \text{int}(\Omega)\), and therefore \(m(\cdot)\) is continuous at \(x = 0\).

**End of Proof**

4 THE APPLICATION OF GPMN

The controller design of nonlinear system with general form has been an important and difficult problem in control theory for a long time. And an often used method in practice is to design a linear controller based on the linearization model near the operation point or equilibrium point.

Linearization model of system (1) near zero of \(x = 0\) can be written as

\[
\dot{x} = Ax + Bu
\]

where

\[
A = \left. \frac{\partial( f(x) + g(x)u) }{\partial x} \right|_{x=0} \quad B = \left. g(x) \right|_{x=0}
\]

Assume \((A, B)\) is controllable, or at least stabilizable. We can design a linear feedback controller \(u = Kx\) such that \(A + BK\) is Hurwitz and the whole linear system has some interesting performance. Simultaneously, we can confirm that the origin of system (1) with the linear controller has an asymptotically stable equilibrium. The linear system theory is so mature that this method can be used in all kinds of real systems. However, the greatest limitation of the linearization controller is that the controller can only be guaranteed to work in some neighborhoods of the equilibrium, which we cannot decide in advance or even predict in most situations.

If we combine the linearized controller design method with the GPMN control, we can solve the problem in part. And in order to verify the feasibility and advantage of our new controller design method, we will give a numerical example.

**Example 1**

\[
\dot{x} = \begin{bmatrix} 
    x_2 \\
    4/3 - 0.2 \cos^2 x_1/(3) \\
    151.57 \end{bmatrix} + \begin{bmatrix} 
    0 \\
    -0.2 \cos x_1 \\
    42.36 \\
    12.96 \\
    42.36 \end{bmatrix} u
\]

\[
V(x) = x^TPx = x^T \begin{bmatrix} 
    151.57 & 42.36 \\
    42.36 & 12.96 \end{bmatrix} x
\]

(8)

System (8) is a pendulum equation from paper \([7]\). And 

\[
\dot{x} = \begin{bmatrix} 
    0 \\
    17.294 \\
    0 \end{bmatrix} + \begin{bmatrix} 
    0 \\
    -0.17647 \\
    0 \end{bmatrix} u
\]

(10)

By using pole placement method, a linear controller can be designed as,

\[
u = 103.7 x_1 + 5.7 x_2
\]

(11)

The GPMN controller with Eq.(11) as a guide function can be denoted as Eq. (13) (in the end of this paper). Solid line in Fig.3 is its time response. It is obvious that the GPMN controller has better convergence performance than that of original PMN controller.
Fig. 4 stability region of example I

Fig. 4 are the phase portrait of the closed loop of linearization controller (thin line) and the stability region of the GPMN $\Omega_{cm}$ (where $cm$ is defined as $\max{\{V \leq -\sigma(x) \text{ for all } x \in \Omega\}}$). From Fig.4, it is obvious that the new controller has larger stability region than the linearization one.

5 CONCLUSION

Based on Freeman’s point wise min-norm controller, a new framework of controller design method, called generalized point wise min-norm controller, is given. The reason we call it a controller design framework is that it can be combined with some other method to obtain new controller with some satisfying performance. And in order to verify its feasibility, we combine it with linearized controller to obtain a new controller which can represents both the local performance of the linearization controller and the large stability region based on CLF.

Furthermore, another work we are doing is to use the GPMN framework in model predictive control to obtain a real time NMPC algorithm with the ensured stability. Also, since the PMN controller is a robust controller design method from [5], and the robust version of GPMN is possible in theory too, which will be the future’s work.

REFERENCES