Nonlinear Predictive Control of Input Constrained System Based on Generalized Pointwise Min-Norm Scheme

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Abstract - Nonlinear Model predictive control (NMPC) suffers from the problems of closed loop instability and computational complexity, which greatly limit the applications of NMPC in real plants involving fast time-varying dynamics. During previous work, the authors have supposed a new real-time NMPC algorithm based on the concept of generalized pointwise min-norm (GPMN) scheme. And in this paper, the new real-time NMPC is generalized to deal with the nonlinear systems with control input constraints. The main contribution of this paper is to find out the analytic form of GPMN controller for input constrained nonlinear systems, and then parameterize it to formulate the real time NMPC controller - called GPMN-enhanced NMPC (GPMN-ENMPC). Finally, in the last section, the simulations with respect to the mobile robot with orthogonal wheel are conducted to verify the feasibility and validity of the new given algorithm.

Index Terms – Control Lyapunov Functions, Constrained Systems, Nonlinear Model Predictive Control

I. INTRODUCTION

Model predictive control (MPC), also called receding horizon control (RHC), has been extensively researched and applied in practice since 1970s because it makes the optimal control theory find its way to the real applications [1]-[4]. However, due to the complicated structure and the required online optimization, some drawbacks of MPC (especially NMPC) algorithms are gradually revealed, such as high computational burden and closed loop instability. And these have greatly limited the applications of NMPC in real plants, especially in the systems involving fast time-varying dynamics[5].

On the one hand, in order to ensure the closed loop stability of NMPC, some extra strategies must be considered, for example, lengthening the predictive horizon[6]; adding terminal state constraints[7]; introducing the concept of CLF[8]. However, it is unavoidable for every one of above approaches to introduce extra computational burden as they usually increase the number of constraints. On the other hand, it is well known that the high computational burden of NMPC, which comes from the process of solving the optimal control problem at each time step, can be reduced by decreasing the number of optimized variables[9]. Unfortunately, the concomitant problem is the deterioration of the stability since it usually relates to the precision of the optimization algorithm.

Therefore, it can be concluded that the stability and computational cost of NMPC algorithms is incompatible with each other. And how to design a stable and fast enough NMPC has been an important research aim that many researchers are pursuing.

Control Lyapunov Functions (CLF) is a new given concept in the 1980s in order to directly make use of the Lyapunov Functions based nonlinear system analysis method to the controller synthesis problem. The concept of CLFs is firstly researched by Artstein in the year of 1983[10], where the equivalent relation between the continuous stabilization of a nonlinear system and the existence of its CLF is firstly given. Although Artstein did not give a specific method to obtain such a continuous stable controller, it still has been a milestone in the nonlinear controller design since several famous formulas of strategies appeared not long after that. Firstly, in 1986, Sontag supposes a ‘universal’ construction method of Artstein’s theorem[11]. Subsequently, Freeman introduced a so called pointwise min-norm control (PMN) based on a known CLF[12]. And recently, in 2004, Curtis[13] proposed another strategy by using the concept ‘satisficing’ combining with CLF to obtain a new controller design method.

The authors have supposed a Generalized Pointwise Min-Norm(GPMN) controller on the basis of Freeman’s PMN controller[4], and introduced it into the NMPC design to partially alleviate the conflict between the closed loop stability and the computational burden[5]. And in this paper, we will further generalize it to deal with the input constrained nonlinear system.

The remainder of this paper is organized as follows. First, section II lists some definitions and former research results on CLFs and GPMN. Second, the analytic form of the GPMN control with respect to input constrained systems is presented in section III. Subsequently, in section IV, the GPMN enhanced NMPC algorithm is supposed. Finally, the simulation results with respect to the planar helicopter system and the conclusions are given respectively in section V and VI.
II. PRELIMINARY CONCEPTS AND RESULTS

In this paper, the following nonlinear input constrained system is considered,
\[ \dot{x} = f(x) + g(x)u \]
\[ u \in U \subset \mathbb{R}^n \tag{1} \]
where \( x \in \mathbb{R}^n \) is the state vector; \( u \) is the control input vector; \( U \) is the control input constraint set; \( f(*) \) and \( g(*) \) are smooth system function with \( f(0) = 0 \). And in this paper, the stabilization problem is researched, i.e., design a controller to drive the state of system (1) to zero point.

Based on Artstein\cite{10}, a CLF of system (1) can be defined as follows,

**Definition 1:**
A CLF of system (1) is a differentiable and positively definite function \( V(x) \) with \( V(0) = 0 \), which is defined on a neighbourhood \( W \) of 0, and such that the following inequality is satisfied,
\[ \inf_{u \in U} [V_x(x)f(x) + V_x(x)g(x)y] < 0 \tag{2} \]
Furthermore, \( V(x) \) is called a global CLF if \( W \) can be chosen \( \mathbb{R}^n \) as \( |x| \rightarrow \infty \).

Given \( c \in \mathbb{R}^n \), \( \Omega_c \) denotes the level set of \( V(x) \) defined as follows
\[ \Omega_c := \{ x \in \mathbb{R}^n : V(x) \leq c \} \tag{3} \]
And \( c_m \) is used to denote the quasi-maximum stabilizable region of system (1), where \( c_m \) is defined as,
\[ c_m := \max \{ c \in \mathbb{R}^n : \Omega_c \subseteq W \} \tag{4} \]
Here, the term quasi-maximum is used because \( \Omega_{c_m} \), the maximum stabilizable region with respect to the CLF- \( V(x) \), is not equivalent to the real stabilizable region of system (1) in most cases.

Freeman’s PMN controller can be denoted as following Eq. (5),
\[ u_{PMN}(x) = \arg \min_{u \in K_v(x)} \|u\| \]
\[ K_v(x) = \{ y \mid V'_v(x)f(x) + V'_v(x)g(x)y \leq -\sigma(x), y \in U \} \tag{5} \]
where \( \sigma(x) \) is a positively definite and continuous function such that \( \sigma(0) = 0 \); \( K_v(x) \) is called the permitted control input set at point \( x \).

In Freeman’s PMN controller (see the sketch of it as Fig.1-a), the control input at each state point is selected as the minimum input value in \( U \) that satisfied the following inequality,
\[ V'_v(x)f(x) + V'_v(x)g(x)y \leq -\sigma(x) \tag{6} \]
Thus, it is called pointwise min-norm controller. Actually, the minimum has little signification except for ensuring the continuity of the controller\cite{12}. The only design parameter of PMN controller is \( \sigma(x) \), however, the selection of \( \sigma(x) \) is restricted since it is greatly related to the stability region of the closed loop system when the CLF is local. Based on this, during the former research, authors introduce a guide function to construct GPMN controller, where \( \sigma(x) \) is only used to ensure the asymptotical stability, and the other desired performances can be flexibly considered through the guide function. And the GPMN controller can be denoted as follows\cite{14},
\[ u_{PMN}(x) = \arg \min_{u \in K_v(x)} \|u - \sigma(x)\| \]
\[ K_v(x) = \{ y \mid V'_v(x)f(x) + V'_v(x)g(x)y \leq -\sigma(x), y \in U \} \tag{7} \]

In controller (7), \( \sigma(x) \), which is continuous with respect to \( x \), is the so called guide function. And in reference [14], it is shown that controller (7) is continuous under some assumptions\cite{14}. The sketch of GPMN controller can be seen in Fig.1-b, where it is clear that the GPMN controller is actually an approach of the guide function under the stability constraints due to CLF. This is just the reason why extra performance requirement can be considered through guide function.

III. ANALYTICAL FORM OF GPMN CONTROLLER

The analytical form of controller (7) is necessary to formulate GPMN enhanced NMPC\cite{15}. However, when the control input constraints have to be dealt with, the analytical
expression of controller (7) might be very complicated or even inexistent. Therefore, in this section we will discuss an easy but general case: a super ball input constraint, i.e.,

$$U = \{(u_1, \cdots, u_n) \mid u_1^2 + \cdots + u_n^2 \leq r^2\} \quad (8)$$

where \((u_1, \cdots, u_n)\) is the input vector.

In order to obtain the analytical expression of controller (7) with input constraints (8), the following 4 steps is proposed,

**Step1:** For every state \(x\), the following equation denotes a super plane in \(\mathbb{R}^n (u \in \mathbb{R}^n)\).

$$V_x f(x) + \sigma(x) + V_x g(x)u = 0 \quad (9)$$

Let \(d\) be the distance from zero to the super plane (9), i.e.,

$$d = \frac{|V_x f(x) + \sigma(x)|}{\sqrt{V_x g(x)g^T(x)V_x}} \quad (10)$$

**Step2:** From Eq. (10), the permitted control input set \(K_{v}(x)\) in controller (7) can be denoted as Fig. 2(a), the right figure (left figure) shows the case that the super plane of (9) intersects (does not intersect) with the super ball (8), and the region filled with the dashed line is \(K_{v}(x)\).

From Fig. 2(a), it can be concluded that, in the case denoted by the left figure, a minimal distance from any point \(p\) to \(K_{v}(x)\) is easily obtained, and the corresponding point in \(K_{v}(x)\) with minimal distance from \(p\) can also be computed (i.e., the point of intersection of the super ball (8) and the beeline connecting the centre of (8) and \(p\)). With respect to the case of the right figure, we use its maximally inscribed super ball \(B'\) to replace \(K_{v}(x)\) (see Fig. 2(b)). Thus, the same way as above can be used to obtain the minimal distance from any point \(p\) to \(B'\) and the corresponding point in \(B'\).

**Step 3:** A new permitted control input sets \(K_{v}(x)\) can be defined as follows,

$$K_{v}(x) = \begin{cases} U & |V_x f(x) + \sigma(x)| \geq r \\ \{u \mid \|u - \gamma(x)\| = R^2(x)\} & \} \end{cases} \quad (11)$$

where

$$\gamma(x) = -\left(\frac{V_x f(x) + \sigma(x)}{2V_x g(x)g^T(x)V_x} + \frac{r}{2\sqrt{V_x g(x)g^T(x)V_x}}\right)g^T(x)V_x$$

$$R(x) = \frac{r - |V_x f(x) + \sigma(x)|}{\sqrt{V_x g(x)g^T(x)V_x}}$$

It is obvious that \(K_{v}(x) \subseteq K_{v}(x)\), which means that the stability of the closed loop can be ensured from the deduction of Proposition 1.

**Step 4:** The analytical form of GPMN controller with input constraint can be described as,

$$
\tilde{u}_{e_{(x)}}(x) = \begin{cases} \frac{\xi(x) - \gamma(x)}{R(x)} & \langle \xi(x) - \gamma(x) \rangle \leq R(x) \\
\langle \xi(x) - \gamma(x) \rangle + \gamma(x) & \text{else}
\end{cases}
$$

From the preceding procedure, it is evident that Eq. (11) is the solution of Eq. (6) with \(K_{v}(x)\) being placed by \(K_{v}(x)\).

**Remark 1:** Indeed, for most control input constraints \(U\), we can always find a maximal inscribed super ball \(B\) of it. And then, the analytic form of controller (7) can be obtained based on the method in this section by using \(B\) to replace \(U\).

FIG. 2 Sketch of the process to build the analytic GPMN controller

**IV. GPMN-ENHANCED NONLINEAR MODEL PREDICTIVE CONTROL**

In order to obtain a stable NMPC with reduced computational burden, we propose to use the GPMN to parameterize the control input sequence in NMPC.

Assuming that \(\xi(x, \theta)\) is a parameterized function mapping from state space to input space, where \(\theta\) is the vector parameters to be designed. The following Eq. (13), called parameterized NMPC, is used to design a real-time NMPC algorithm with ensured closed loop stability.

Controller (13) is different from the normal NMPC\(^1\) where one try to optimize the continuous control profile of \(u\), and the NMPC of (13) tries to achieve the optimal performance by the optimization of the parameter vector \(\theta\). Thus, the computational cost will be mainly dominated by the number of the parameters in \(\theta\).

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1. [1]
\[ u^* = \mathcal{G}(x, \theta') \]
\[ \theta' = \arg \min_{\theta'; \theta} J(x, \theta) \]
\[ J(x, \theta) = \int_{t}^{T} (x^T P x + u^T Q u) d\tau \]
\[ s.t. \quad \dot{x} = f(x) + g(x) \mathcal{G}(x, \theta) \]
\[ \mathcal{G}(x, \theta) \in U, \forall t \in [t, t + T] \]

The following proposition shows that the closed loop stability of controller (13) can be ensured if \( \mathcal{G}(x, \theta) \) in Eq. (13) is replaced by the GPMN controller (7).

**Proposition 1:**
Assuming \( V(x) \) is a CLF of system (1), \( \Omega \) is the stability region of \( V(x) \), then the following controller (13), called GPMN-Enhanced NMPC (GPMN-ENMPC), with a defined \( \mathcal{G}(x, \theta) \) is stable in \( \Omega \),
\[ \mathcal{G}(x, \theta) = u_{(x, \theta)}(x, \theta) = \arg \min_{u \in u(x)} \|u - \xi(x, \theta)\| \]
where \( u_{(x, \theta)}(x, \theta) \) is the GPMN control and \( \xi(x, \theta) \) the guide function.

Furthermore, if \( V(x) \) is a global CLF, the controller (13)-(14) is stable over \( \mathbb{R}^n \).

**Proof:**
At any time instant \( t \), by assuming that \( \theta' \) is the optimal parameter of the GPMN-ENMPC algorithm, the control input at \( t \) is the GPMN control \( u_{(x, \theta')} \). Thus, from reference [14], \( u_{(x, \theta')} \) can guarantee a negative definite \( \dot{V}(x) \). This means that the GPMN-ENMPC can make \( \dot{V}(x) \) along with the trajectory of system (1) be negative in any time instant. Thus, the closed loop stability of controller (13)-(14) can be shown easily by selecting \( V(x) \) as a Lyapunov function.

**Remark 2:** From Proposition II, the closed loop stability of GPMN-ENMPC algorithm can be ensured regardless of the structure of \( \xi(x, \theta) \) and \( \theta \). This is the key point for us to obtain the flexibility during the process of controller design by regulating the structure of guide function or changing the number of unknown parameter \( \theta \) to pursue interested closed loop performance.

Theoretically, \( \xi(x, \theta) \) in (14) can be selected in any form since it does not influence the closed loop stability, and a stable GPMN controller can always be guaranteed for any \( \theta \) by the GPMN control of (7). However, just like the basis function in Predictive Function Control (PFC) algorithm, \( \xi(x, \theta) \) will definitely influence the performance of the GPMN-ENMPC. Thus, we propose to use Bellman’s principle of optimality to design \( \xi(x, \theta) \).

In general, \( J(x, u) \) in NMPC has the following quadratic form,
\[ J(x, u) = \int_{t}^{T} (x^T P x + u^T Q u) d\tau \]
Thus, the following optimal control law based on Bellman’s principle of optimality \(^{16} \) can be obtained,
\[ u^* = \frac{1}{2} (Q^{-1})^T g^T (x) \frac{\partial J^*}{\partial x} \]
where \( J^* \) is the optimal value function of \( J(x, u) \).

Unfortunately, in most cases, it is impossible to obtain such an optimal value function.

Simultaneously, from the Stone-Weierstrass theorem \(^{17} \), every continuous function defined in a bounded set can be uniformly approximated by a polynomial function as follows,
\[ B_k^V (x_1, \ldots, x_n) = \sum_{\nu_1, \ldots, \nu_n \geq 0, \nu_1 + \cdots + \nu_n \leq k} p_{k, \nu_1, \ldots, \nu_n} (x_1, \ldots, x_n) \quad \text{(17)} \]
where
\[ p_{k, \nu_1, \ldots, \nu_n} (x_1, \ldots, x_n) = \left( \begin{array}{c} k \\ \nu_1, \ldots, \nu_n \end{array} \right) x_1^{\nu_1} \cdots x_n^{\nu_n} \left( 1 - x_1 - \cdots - x_n \right)^{k-\nu_1-\cdots-\nu_n} \]
and
\[ \lim_{k \to \infty} B_k^V (x_1, \ldots, x_n) = J^* (x_1, \ldots, x_n) \quad \text{(19)} \]

Thus, although the optimal function of \( J^* \) is difficult to be obtained, it can be approached by using Eq. (17). And by selecting polynomial parameters optimally, a ‘quasi-optimal’ function, which is closed to \( J^* \), can be obtained. This means, \( \xi(x, \theta) \) can be selected as Bernstein polynomial with pre-designed order, i.e.,
\[ \xi(x, \theta) \triangleq \sum_{\nu_1, \ldots, \nu_n \geq 0, \nu_1 + \cdots + \nu_n \leq k} \theta_{k, \nu_1, \ldots, \nu_n} p_{k, \nu_1, \ldots, \nu_n} (x_1, \ldots, x_n) \quad \text{(20)} \]
where \( \nu_1, \ldots, \nu_n \geq 0 \) and \( \nu_1 + \cdots + \nu_n \leq k \) are the parameters to be optimized, \( k \) is the order of the Bernstein polynomial.

**Remark 3:** The selection of \( \xi(x, \theta) \) as Eq. (20) provides a feasible way to complete the GPMN-ENMPC. In this way, the computation cost is controllable. Namely, one can select the order of \( n \) to meet the CPU capability of a specific real system. And, the selection of \( n \) does not influence the closed loop stability, which has already guaranteed by the GPMN scheme. But there still exists a trade-off between computation cost and the optimal performance which is determined by \( \xi(x, \theta) \).

V. SIMULATIONS

In order to test the ability of GPMN-ENMPC in handling input constraints, another simulation with respect to the mobile robot with orthogonal wheel assemblies dynamics \(^{18} \) is conducted in this subsection. The system model is as follows,
\[ \dot{x} = f(x) + g(x) u \quad \text{(21)} \]
where \( f(x) \) and \( g(x) \) is as Eq. (22).
\[ f(x) = \begin{bmatrix} x_2 \\ -2.3684x_4x_6 - 0.5921x_3 \\ x_4 \\ 2.3684x_2x_6 - 0.5921x_4 \\ x_6 \\ -0.2602x_5 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0.8772(-\sqrt{3}\sin x_5 - \cos x_5) \\ 0.8772*2\cos x_5 \\ 0.8772(-\sqrt{3}\cos x_5 - \sin x_5) \\ 0 \end{bmatrix} \]

\[ x_1 \text{ and } x_2 \text{ are the } x, y \text{ position of the robot system; } x_3 \text{ and } x_4 \text{ are the correspond velocity; } x_5 \text{ and } x_6 \text{ are the yaw angle and angular velocity; } u_1, u_2, \text{ and } u_3 \text{ are the torque inputs.} \]

Suppose that the control input is restricted in the following closed set,

\[ U = \{(u_1, u_2, u_3) \mid \sqrt{u_1^2 + u_2^2 + u_3^2} \leq 20\} \quad (23) \]

It is not difficult to show that system (21) is feedback linearizable, and thus we can obtain a CLF of it,

\[ V(x) = x^TPx \quad (24) \]

where

\[ P = \begin{bmatrix} 1.125 & 0.125 & 0 & 0 & 0 & 0 \\ 0.125 & 0.156 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.125 & 0.125 & 0 & 0 \\ 0 & 0 & 0.125 & 0.156 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.125 & 0.125 \\ 0 & 0 & 0 & 0 & 0.125 & 1.156 \end{bmatrix} \]

Designing the following cost function \( J(x) \) and \( \sigma(x) \),

\[ J(x) = \int_{-\infty}^{\infty} \left( 3x_1^2 + 3x_2^2 + 3x_3^2 + x_4^2 + x_5^2 + x_6^2 + 5u_1^2 + 5u_2^2 + 5u_3^2 \right) dt + v\theta^T(k-1)\theta(k-1) \quad (25) \]

\[ \sigma(x) = 0.1(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) \]

where the added item of \( v\theta^T(k-1)\theta(k-1) \) in \( J(x) \) is used to smooth the computational results of GPMN-ENMPC between two neighboured optimization.

The integral in cost function is approached by the following equation,

\[ \int_{-\infty}^{\infty} h(t)dt = \sum_{i=1}^{\text{int}[T/\Delta T]} h(t + i\Delta T)\Delta T \quad (26) \]

where the integral constant \( \Delta T \) is selected as 0.1s. And the Genetic Algorithm (GA function) in MATLAB toolbox is used as nonlinear optimization solver.

Furthermore, in order to save the computation cost, we will reduce the frequency of the optimization in this simulation, i.e., optimization process is conducted at every 0.1s while the controller of (12) is calculated at every 0.002s. And this will not influence the closed loop stability.

The time response with initial state (10; 5; -10; -5; 1; 0) is shown in Fig.3. And the maximum of \( \sqrt{u_1^2 + u_2^2 + u_3^2} \) during the simulation is 400, and which appears at initial time instant. From Fig.3 it is clear that the GPMN-ENMPC has the ability to deal with the input constraints and drive the robot system to its desired state point.
The parameterized NMPC was used to formulate the new algorithm in [15], the main problem of the analytical expression of the GPMN controller with control input constraints was solved in this paper, and which, combined to the parameterized NMPC, was used to formulate the new proposed controller. In the new proposed algorithm, the optimizing variables had changed from the control input profile in classical NMPC algorithms into the parameters of GPMN controller. Thus, the dependence between the closed loop stability and the computational burden were weakened, and the conflict between them was also alleviated. Finally, the simulation results showed that the new proposed algorithm is feasible and valid.

VI. CONCLUSIONS

In this paper, a nonlinear model predictive control enhanced by generalized pointwise min-norm (GPMN) scheme, a nonlinear controller based on CLF, was proposed with respect to the input constrained system to alleviate the conflict between the computational burden and the closed loop stability in classical NMPC algorithms. This paper was based on the authors' previous work in reference [15]. Different with the algorithm in [15], the main problem of the analytical expression of the GPMN controller with control input constraints was solved in this paper, and which, combined to the parameterized NMPC, was used to formulate the new proposed controller. In the new proposed algorithm, the optimizing variables had changed from the control input profile in classical NMPC algorithms into the parameters of GPMN controller. Thus, the dependence between the closed loop stability and the computational burden were weakened, and the conflict between them was also alleviated. Finally, the simulation results showed that the new proposed algorithm is feasible and valid.

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