Receding Horizon Optimization Enhanced Robust GPMN Control for A Certain Kind of Nonlinear Systems

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Abstract—In this paper, a new robust and optimization based nonlinear controller with respect to a certain kind of nonlinear disturbed systems is proposed based on control Lyapunov functions (CLF) concept and receding horizon control (RHC) scheme. This new controller can be divided into two components, one is the robust control law with measurable disturbance information; another is the robust receding horizon algorithm which is designed by combining the new given robust generalized pointwise min-norm scheme (RGPMN) to the traditional RHC. The most attractions of the new algorithm include that, (1) the closed loop BIBO stability can always be ensured; (2) the sub-optimality can be obtainable from the receding horizon strategy; and (3) the computational burden introduced by the receding horizon optimization of the final controller is regulable in different applications.

I. INTRODUCTION

In almost all real systems, optimality is one of the key performances that people try to pursue when designing controller, while the uncertainties existing in the models of the interested system greatly increase the difficulty of it, especially for the controller design of nonlinear systems. Thus, during past decades, optimal/sub-optimal control of uncertainty systems has been a top important topic in control theory and attained much attention.

On one hand, nonlinear robust control, i.e., design controller that is insensitive to the uncertainties in system’s model, has achieved great development in recent years. And many tools are introduced to obtain the robustness of the closed loop systems, including the differential geometry, the heuristic ideas, and the $H_{\infty}$ control strategy.

Differential geometry is a powerful tool to design nonlinear robust controller, on which several remarkable nonlinear robust controllers based can be obtained, see references [1] and [2]. And, the heuristic controller design methods, including neural network control and fuzzy control, which tries to learn experiences dealing of uncertainties from human being, have also been extensively used in many researches [3][4]. Finally, as far as the external disturbances are concerned, $H_{\infty}$ control is a good choice[5]. Unfortunately, the common drawback of all these above robust control methods, i.e., the optimization is difficult to be considered with respect to some pre-defined complicated performance index, makes the nonlinear robust and optimal control still an open problem in control theory.

On the other hand, optimal control theory has been developed for more than 50 years, and great deals of achievements have been obtained. However, it is well known that the optimal controllers can hardly be used in reality, because they are inherent open loop algorithm and present poor robustness[6].

Receding horizon control (RHC), or Model based predictive control (MPC), is a successful strategy to implement the optimization in control engineering. And it has been the most-often-used control strategy in industry except for the PID controller[7][8]. Although the robustness is an initially desired performance of the RHC[24], it has been proved that the robustness of normal RHC is not as good as being expected[9]. Thus, robust RHC (RRHC) has attracted much research interesting in recent years, and some important results have been published[10][11]. Up to now, there still exist some difficulties during designing the RRHC, such as the accurate prediction problem of the future states[12], the closed loop instability[13][14], and the huge computational burden[15][16]. These, naturally, will be much heavier if the system model includes nonlinearities, i.e., the robust nonlinear RHC (RNRHC).

In this paper, with respect to a certain kind of nonlinear systems, a new robust optimization based nonlinear controller, which can partially solve the preceding difficulties, will be introduced by using the GPMN scheme researched by the authors of this paper in reference [17].

II. PROBLEM STATEMENT

The nonlinear system under consideration in this paper is as follows,

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u + l(x)\omega \\
y &= h(x)
\end{align*}
\]

(1)

where $x \in \mathbb{R}^n$ is state vector; $u \in \mathbb{R}^m$ is control input; $\omega \in \mathbb{R}^r$ is external disturbance; $f(\cdot), f(0) = 0$, $g(\cdot), l(\cdot)$ and $h(\cdot)$ are all pre-defined nonlinear smooth functions.

Further, we suppose system (1) satisfies the following assumptions,

Assumption I:

System (1) is nominal static feedback linearizable, i.e., there exists a state feedback controller $u = k(x)$ such that (1) can be transformed into a linear system with $\omega = 0$. 

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Assumption II:
The disturbances of system (1) are partially obtainable, i.e., the variables \( \omega \) can be used to construct controller.

The main interest of this paper is to design a robust and optimization based controller to stabilize system (1) under the assumption I and II.

Remark 1: Assumption II is reasonable because the disturbances of system (1) are partially obtainable, i.e., the variables \( \omega \) can be used to construct controller. However, the higher order derivative of the disturbances with respective to time is often difficult to be obtained due to the heavy additive noise. Thus, the disturbances are often 'Partially Obtainable'.

III. H\(_{\infty}\) Control With Partially Known Disturbances

If the disturbances are partially obtainable, it is natural that the better disturbance attenuation performance can be desired. In this section, we will firstly design an H\(_{\infty}\) robust controller with partially known disturbance information. More detailed description of this section can be found in reference [18].

Based on Assumption I, system (1) is static feedback linearizable, that means (1) can be transformed into the following equations through some coordination transformation matrix,

\[
\begin{align*}
\dot{z}_1 &= z_2 + F_1(z)\Delta \\
\vdots \\
\dot{z}_n &= f(z) + g_1(z)u + F_n(z)\Delta \\
y &= z_1
\end{align*}
\] (2)

where \( z = [z_1, \ldots, z_n]^T \) is the new defined state variable.

An H\(_{\infty}\) robust controller for system (1) can be obtained as the following theorem.

Theorem 1

Considering system (2), if there exists a control \( u = u(z) \) and a radially unbounded function \( V(x) \) to satisfy the following inequality,

\[
\begin{align*}
\sum_{i=1}^{n-1} V_{z_i} z_{i+1} + V_z [f(z) + g(z)u(z)] + \frac{2}{\gamma^2} V_z \left[ \begin{array}{c} F_1^T(z) \\
F_2^T(z) \\
\vdots \\
F_n^T(z) \end{array} \right] \left[ \begin{array}{c} F_1^T(z) \\
F_2^T(z) \\
\vdots \\
F_n^T(z) \end{array} \right]^T V_z + z_1^2 &\leq 0
\end{align*}
\]

By introducing a new state vector \( \rho \), controller \( u = g_1(z)[f(z) + g(z)u(z)] + \sum_{i=1}^{n-1} F_i(z, \rho, \ldots, \rho^{(i-1)})\rho^{(i)} + \left[ F_n(z) \right]^{-1} \times \frac{2}{\gamma^2} F_n(z)^T \left[ \begin{array}{c} F_1(z) \\
F_2(z) \\
\vdots \\
F_n(z) \end{array} \right] [F_1(z) - F_1(z, \rho) \\
F_2(z) - F_2(z, \rho, \rho) \\
\vdots \\
F_n(z) - F_n(z, \rho, \ldots, \rho^{(n-1)})]_{\rho} \]

where \( z = \left[ z_1, \ldots, z_{n-1} \right] \) is the new defined state variable.

An H\(_{\infty}\) robust controller for system (1) can be obtained as the following theorem.

Proof of Theorem 1

Define new variables,

\[
\begin{align*}
\bar{z}_1 &= z_1 \\
\bar{z}_2 &= z_2 - F_1(z)\rho \\
\vdots \\
\bar{z}_n &= \sum_{i=1}^{n-1} F_i(z)\rho^{(i-1)} \\
y &= \bar{z}_1
\end{align*}
\]

Then, system (2) can be rewritten as

\[
\begin{align*}
\ddot{\bar{z}}_1 &= \bar{z}_2 + \bar{F}_1(z, \rho)(\Delta + \rho) \\
\ddot{\bar{z}}_2 &= \bar{z}_3 + \bar{F}_2(z, \rho, \rho)(\Delta + \rho) \\
\vdots \\
\ddot{\bar{z}}_n &= \bar{f}(z) + g(z)u - \sum_{i=1}^{n-1} \bar{F}_i(z, \rho, \ldots, \rho^{(i-1)})\rho^{(i)} + \bar{F}_n(z, \rho, \ldots, \rho^{(n-1)})(\Delta + \rho) \\
y &= \bar{z}_1
\end{align*}
\]

where

\[
\begin{align*}
\bar{F}_j(z, \rho, \rho^{(j-1)}) &\triangleq F_j(z)\big|_{z_1=z_1^j(z, \rho, \ldots, \rho^{(j-1)})}
\end{align*}
\]

Let

\[
\begin{align*}
\bar{V}(z) &= V(z)\big|_{z_1=\bar{z}_1} \\
\end{align*}
\]

Computing the HJI equation\(^{[20]}\) of system (6) with respect to \( \bar{V}(z) \), we have,

\[
\sum_{i=1}^{n-1} \frac{2}{\gamma^2} F_i(z) \left[ F_i(z, \rho, \ldots, \rho^{(i-1)}) \right] + \left[ F_n(z, \rho, \ldots, \rho^{(n-1)}) \right] F_n(z, \rho, \ldots, \rho^{(n-1)})^T + z_i^2 \leq 0
\]

Thus, combine controller (4) and Eq. (8), we have,

\[
\begin{align*}
\sum_{i=1}^{n-1} \frac{2}{\gamma^2} F_i(z) \left[ F_i(z, \rho, \ldots, \rho^{(i-1)}) \right] + \left[ F_n(z, \rho, \ldots, \rho^{(n-1)}) \right] F_n(z, \rho, \ldots, \rho^{(n-1)})^T + z_i^2
\end{align*}
\]

where

\[
\begin{align*}
\bar{z} &= \left[ \bar{z}_1, \ldots, \bar{z}_n \right]^T
\end{align*}
\]

Using the fact that the disturbances of system (1) are partially obtainable, i.e., the variables \( \omega \) can be used to construct controller. However, the higher order derivative of the disturbances with respective to time is often difficult to be obtained due to the heavy additive noise. Thus, the disturbances are often 'Partially Obtainable'.
Thus, based on theorem 5.5 in reference [20], controller (4) can make system (2) finite gain $L_2$ stable from $\Delta + \rho$ to $y$, and the $L_2$ gain is less than or equal to $\gamma$.

Except for the $H_{\infty}$ controller, $\rho$ can be used to further attenuate the disturbances which are partially obtainable from assumption II by the following equation,

$$ \rho(s) = \frac{B(s)}{A(s)} \Delta(s) $$

(10)

where $s$ is the Laplace operator in frequency domain. Thus, the new external disturbances $\Delta + \rho$ can be denoted as,

$$ \Delta(s) + \rho(s) = \frac{A(s)}{A(s)} \Delta(s) $$

(11)

From Eq. (11), by properly designing $A(s)$ and $B(s)$, the influence of disturbance $\Delta$ on outputs can be attenuated effectively.

IV. ROBUST GPMN CONTROL BASED ON CONTROL LYAPUNOV FUNCTIONS

In many applications, performance except for the robustness is usually necessary, such as optimality. Thus, controller (4) must be improved to satisfy different performance requirement. In this section, by using the concept of Control Lyapunov Functions (CLF), we will design a new control strategy to make the robust controller in section III convenient to be improved for different performance index.

CLF is a successful attempt of directly using Lyapunov based stability analysis principle into the design of stable controllers. Since being proposed in 1983 by Artstein[21], CLF based nonlinear controller design method has obtained great development[22][23]. In this section, we will firstly give out the following generalized version of traditional concept of CLF in [21],

**Definition I.**

A $C^1$ and positively definite function $V(x)$ is called an local $H_\infty$-CLF of system (1) in $\Omega_c$, ($\Omega_c := \{x \in \mathbb{R}^n; V(x) \leq c\}$) if there exists a positive constant $c$ such that the following inequality

$$ \inf_{u,c} [V_i(f(x) + g(x)u) + \frac{1}{2}V_i(h(x))V_i^T + \frac{1}{2}h^T(x)h(x)] \leq 0 $$

(12)

is satisfied for all $x \in \Omega_c$.

Furthermore, $V(x)$ is called a global $H_\infty$-CLF if $c$ can be chosen $+\infty$ with $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

From Definition I, a CLF indicates that an permitted control set, i.e., the set being composed of the inputs satisfying inequality (12), can be found and used at every state point to form the stable controller. That means, in order to obtain a stable controller, what one need to do next is just to find a strategy on how to select a control action in the permitted control set.

In reference [17], based on the concept of $H_\infty$-CLF, we have proposed the following $H_\infty$ generalized pointwise min-norm ($H_\infty$GPMN) controller,

$$ u_{\Delta} = \arg \min_{u \in K_i} \| u - \xi(x) \| $$

$$ K_i = \{ u : V_i(f(x) + g(x)u) + \frac{1}{2}V_i(h(x))V_i^T + \frac{1}{2}h^T(x)h(x) \leq -\sigma(x) \} $$

(13)

where $\sigma(x)$ is a positively definite function for ensuring the asymptotical stability of the closed loop; $\xi(x)$, called guide function, is a new introducing function to be designed. Furthermore, if the control inputs are unconstrained, controller (13) can also be explicitly denoted as the following analytic form,

$$ u_{\Delta} = \begin{cases} 
\xi - \frac{1}{2\gamma^2}V_i(f(x) + g(x)u) + \frac{1}{2}h^T(x)h(x) \leq -\sigma(x) 
\end{cases} $$

(14)

IV. ROBUST NONLINEAR RHC

Predictive control is a useful strategy to bring the optimality. In this section, we will discuss how to use it to improve the optimization ability of the controller given in section II and III, and obtain a new controller called Robust Nonlinear RHC (NRHC).

A. Normal NRHC Formulation

Without considering the external disturbances, the continuous version NRHC of system (1) can be denoted as,

$$ u = \arg \min_{u \in c} J(x,u) $$

$$ J(x,u) = \int_{t}^{t+T} R(x(\tau),y(\tau),u(\tau))d\tau + \phi(x(t+T)) $$

(15)

s.t. $\dot{x} = f(x) + g(x)u$

$$ x(t+T) \in X_f $$

where $R(\cdot,\cdot,\cdot)$ is a continuous and positively definite cost function with $R(0,0,0) = 0$; $t$ and $T$ are respectively current time instant and predictive horizon; $\phi(x(t+T))$ and $X_f$ are respectively the terminal cost function and the terminal constraint set to ensure the closed loop stability[2].

As far as the external disturbances are concerned, nominal model based Nonlinear RHC (NRHC), where the prediction
is made through a nominal certain system model, is an often used strategy in reality. And the formulation of it is very similar to Eq. (15) except that $X_f$ and $\varphi(x(t+T))$.

However, for disturbed nonlinear system like Eq. (1), NRHC algorithm of Eq. (15) can hardly be used in real applications due to the huge computational burden and weak robustness. Thus, in the next subsection, we will combine the NRHC strategy with the robust controller in section III and section IV to overcome the drawbacks originated from both nominal NRHC algorithm and the robust controller (4) and (14).

**B. Parameterized RNRHC Algorithm**

In this section, we will design the new parameterized RNRHC algorithm based on controller (4) and (14). The detailed structure of the new given controller is as Fig. 1.

Figure 1: Structure of new designed RNRHC controller

From Fig.1, in order to decrease the computational burden, only a group of optimal parameters, being introduced into the $H_\infty$ GPMN controller, needs to be computed during the optimization process in the NRHC algorithm.

Eq. (16) is the new designed RNRHC algorithm, called E-RNRHC. Compared to Eq.(15), it is easy to find out that the control input in the new RNRHC algorithm has a pre-defined structure given in section III and section IV.

$$u^* = u^*_{H\infty}(x, \theta^*)$$

$$\theta^* = \arg \min_{\theta} J(x,u)$$

$$J(x,u) = \int_{-T}^{T} l(x(\tau), u(\tau))d\tau$$

$s.t. \quad \dot{x} = f(x) + g(x)u$

$$u(\tau) = u^*_{H\infty}(x, \theta)$$

Thus, we can obtain the following two conclusions,

1. The optimizing variables are replaced by some structural parameters $\theta$, thus, the computational cost of the new controller can be freely regulated by reducing the number of $\theta$;

2. The optimizing variables in algorithm (16) is the parameters of RGPMN, which means that any group of parameters can be used to form a RGPMN controller that can stabilize the system by the concept of CLF. This is just the reason why we can flexibly trade off between the computational cost and the optimality without destroying the closed loop stability.

VI. PRACTICAL CONSIDERING

RNRHC algorithm can be divided into two processes, including the implementation process and the optimization process, and the sketch of it can be shown as Fig.2.

![Fig.2 The process of ERNRHC](image)

The implementation process of Fig.2 is composed of 3 parts as the sub-figure surrounded by the dashed line. From Fig.2, the implementation process and the optimization process are independent. In implementation process, the RGPMN scheme is used to ensure the closed loop stability, and in the optimization process, the optimization process is responsible to improving the optimality of the controller. And the interaction of the two processes is realized through the optimized parameter $\theta^*$ (from optimization process to implementation process) and the measured states (form implementation process to optimization process).

A. TIME INTERVAL BETWEEN TWO NEIGHBORING OPTIMIZING PROCESS

The sample time in the control of mechatronic system is often several milliseconds, which is very challenging to implement complicated algorithm as NRHC. Fortunately, the optimization process will end up with a group of parameters which are used to form a stable RGPMN controller, and the optimization process itself does not influence the closed loop stability at all. Thus, theoretically, every group of optimized parameters can be used for several sample intervals without destroying the closed loop stability.

![Fig.3 Scheduling of ERNRHC](image)
Fig. 3 denotes the scheduling of RNRHC algorithm. In Fig. 3, $t$ is the current time instant; $T$ is the prediction horizon; $T_s$ is the sample time of the RGPMN controller; and $T_i$ is the implementation time of every optimal parameter $\theta_i^*(t)$, i.e., the same parameter $\theta_i^*$ is used to implement the GPMN controller from time $t$ to time $t+T_i$.

B. Numerical Integrator

How to predict the future’s behavior is very important during the implementation of any kind of NRHC algorithms. In most applications, the NRHC algorithm is realized by computers. Thus, for the continuous systems, it will be difficult and time consuming if some accurate but complicated numerical integration methods are used, such as Newton-Cotes integration and Gaussian quadratures, etc. In this paper, we will discretize the continuous system (1) as follows,

$$x(kT_o + T_o) = f(x(kT_o))T_o + g(x(kT_o))u(kT_o)T_o$$

(17)

where $T_o$ is the discrete sample time.

C. Index Function

By replacing $x(kT_o)$ with $x(k)$ in the following sections. The index function can be designed as follows,

$$J(x(k_s), \theta) = \theta^T Z \theta^* + \sum_{i=k_s}^{k_o+N} x^T(i)Qx(i) + k^T(i)Rk(x(i), \theta)$$

(18)

where $k_s$ denotes the current time instant; $N$ is the predictiv horizon with $N=\text{int}(T/T_o)$; $\theta$ is the parameter vector to be optimized at current time instant; and $\theta^*$ is the last optimization result; $Q$, $Z$, $R$ are constant matrix with $Q>0$, $Z>0$, and $R>0$.

The new designed item $\theta^T Z \theta^*$ is used to reduce the difference between two neighboring optimized parameter vector, and improve the smoothness of the optimized control inputs $u$.

VII. SIMULATION RESULTS

In this section, a simulation will be given to verify the feasibility of the proposed algorithm with respect to the following planar dynamic model of helicopter,

$$\begin{align*}
\dot{x} &= -9.8 \cos \phi \sin \theta + \Delta_1 \\
\dot{y} &= 9.8 \sin \phi + \Delta_2 \\
\dot{\phi} &= 0.05 \dot{\theta} \sin \phi \cos \phi + \dot{\phi} \tan \theta (0.5 + 0.05 \cos^2 \phi) + 0.07 \cos \phi \tan \theta + L + M \sin \phi \tan \theta + \Delta_3 \\
\dot{\theta} &= -0.05 \dot{\phi} \dot{\sin \phi} \cos \phi - 0.07 \sin \phi + M \cos \phi + \Delta_4
\end{align*}$$

(19)

where $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$ are all the external disturbances, and are selected as following values,

$$\begin{align*}
\Delta_1 &= 3; & \Delta_2 &= 3 \\
\Delta_3 &= 10 \sin(0.5t) \\
\Delta_4 &= 10 \sin(0.5t)
\end{align*}$$

Firstly, design an $\mathcal{H}_\infty$-CLF of system (19) by using the feedback linearization method,

$$V = X^T \hat{P} X$$

(20)

where,

$$X = [x, \dot{x}, \ddot{x}, x, \dot{x}, \ddot{x}, \gamma, \dot{\gamma}, \ddot{\gamma}]^T$$

$$\begin{bmatrix}
14.48 & 11.45 & 3.99 & 0.74 & 0 & 0 & 0 & 0 \\
11.45 & 9.77 & 3.44 & 0.66 & 0 & 0 & 0 & 0 \\
3.99 & 3.44 & 1.28 & 0.24 & 0 & 0 & 0 & 0 \\
0.74 & 0.66 & 0.24 & 0.05 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 14.48 & 11.45 & 3.99 & 0.74 \\
0 & 0 & 0 & 0 & 11.45 & 9.77 & 3.44 & 0.66 \\
0 & 0 & 0 & 0 & 3.99 & 3.44 & 1.28 & 0.24 \\
0 & 0 & 0 & 0 & 0.74 & 0.66 & 0.24 & 0.05 \\
\end{bmatrix}$$

Thus, the robust predictive controller can be designed as Eq. (4), (14) and (16) with the following parameters,

$$a(x) = X^T X$$

$$J = \theta^T T_0 \theta^* + \sum_{i=1}^{N} [x^T(iT_o) Px(iT_o) + u^T(iT_o) Qu(iT_o) T_o]$$

(21)

The optimality performance of RNRHC, computed from Eq. (21), is about 3280, and the feedback linearization controller is about 5741. That shows that the RNRHC has better optimality than the feedback linearization controller.
In this paper, a new robust control strategy with the measurable disturbance information is firstly given with respect to the feedback linearizable nonlinear disturbed systems. Then, a robust generalized pointwise min-norm controller is designed based on control Lyapunov functions and combined with the new-given controller to obtain a robust model predictive control (RNRHC) controller which can make some other performance requirement easily be considered. Subsequently, parameterized receding horizon strategy is introduced to the new RNRHC controller to form a new RNRHC controller, which the closed loop performance can be further optimized by. The new proposed algorithm can not only ensure the closed loop stability, but also freely reduce the computational burden at the price of deteriorating the optimality to some extent. Finally, the simulations with respect to the planar dynamics of a model helicopter show the feasibility and validity of the new robust predictive control algorithm.

REFERENCES

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