Variance Active Contour

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Abstract: A simple gradient flow called variance active contour is proposed by defining a new inner product on the set of perturbations of a curve. It favors global translation, but is not restricted to it, like $H^1$ active contour defined with a Sobolev-type inner product. In contrast to $H^1$ active contour generated by convolution of $H^2$ active contour defined using the $L^2$-type inner product and certain kernel functional, variance active contour needs no convolution and is a weighting sum of $H^2$ active contour and corresponding average gradient flow. Moreover, variance active contour can be realized exactly using level set methods without polygon extraction which is necessary for $H^1$ active contour and much more difficult for surfaces. Thus, Variance active contour can be implemented much faster and easier than $H^1$ active contour. We also give comparisons in space and frequency domains and provide experiment results for $H^2$, $H^1$, and variance active contours.

Keywords: variance active contour, Sobolev active contours, inner products, gradient flows, Gâteaux derivative

方差主动轮廓

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摘要: 本文通过定义在曲线抗干扰集合上的新内积空间导出了简单梯度流，叫做方差主动轮廓。方差主动轮廓与 $H^1$ 主动轮廓一样具有平移优先性，但并不局限于平移演化。方差主动轮廓是将 $H^2$ 主动轮廓与其对应的平均梯度流加权求和得到，而 $H^1$ 主动轮廓是通过 $H^2$ 主动轮廓与特定类型的核函数进行卷积得到。当用水平集方法精确实现 $H^1$ 主动轮廓时需要提取出多边形，而且对不规则曲面提取更困难，而方差主动轮廓无需此操作。因此方差主动轮廓实现时更简单和快速。最后我们给出了 $H^2$, $H^1$ 和方差主动轮廓在频域与时域的分析和实验结果比较。

关键词: 方差主动轮廓, Sobolev 主动轮廓, 内积, 梯度流, 加特微分

1 Introduction

Active contour model, also known as snake model, was pioneered in 1987 by Kass et al. in[1] for image segmentation via driving an initial contour toward a desired object edge with a PDE deduced by minimizing an energy functional. But it has many limitations. For example, it is hard to undergo topological change, sensitive to initial contour placement, and dependent on curve parameterization. In 1993, geodesic active contour[2], formulated as a weighting Euclidean arc length using a edge-stopped potential functional, was proposed by Casselles et al. in a level set framework[3]. It is independent of curve parameterization and can easily handle curve topological changes. Those models mentioned above are edge-based[2, 4]. In most cases, they are less robust for image segmentation than region-based active contour models[5-7] utilizing certain image global region statistical information, which partition a given image into statistically distinct regions. The Mumford-Shah (M-S)[8] model proposed by Mumford and Shah in 1989 can be considered as a whole image segmentation functional framework for edge and region based active contour models. The paradigm to combine the edge and region information for active contours was presented in[9].

In order to obtain desirable segmentation results, an important strategy for active contours research is combining certain prior knowledge with object information (gradient and region information etc) to deal with images with insufficient information. Shape and topology of objects
to be segmented are important prior information. Some paradigms making use of prior shape and topology information can be found in [10–14]. For more recent developments about active contours, we refer the readers to [15, 16].

In order to entitle curve evolution to certain desirable features, prior information on the deformation field of evolving contours can also be used. Clearly, the gradient flow driving curve evolution derived by minimizing its corresponding energy functional is directly relative to the choice of the inner product on the deformation field defined on a manifold of curves. Thus, a prior information on the deformation field can be embodied directly by the choice of the inner product [17]. However, before the appearances of [17, 18], the authors focused on energy functional building, while they ignored the relation between gradient flows and choices of inner products completely and considered that the deformation field is ruled by the $L^2$-type inner product called $H^0$ active contour in ref [18]. Due to the arbitrary evolution of $H^0$ active contour which results in many undesirable features [18], Sundaramoorthi et al. used the $H^1$-type inner product to define the gradient flow for active contour called $H^1$ active contour, which has many advantages over $H^0$ active contour, for example, favorable global translations, better regularity properties, and desirable ability of coarse to fine motion, which are more suitable for image segmentation and object tracking.

In this paper, a new inner product is defined which induces a new active contour called variance active contour favoring global translations. However, $H^1$ active contour is a convolution of $H^0$ active contour and a kernel functional, which is derived by solving an ODE equation, while variance active contour is a simple weighting sum of $H^0$ active contour and corresponding average gradient flow, derived by solving a simple algebra equation. In contrast to $H^1$ active contour, variance active contour can be realized exactly using level set methods without polygon extraction and implemented easily and fast.

## 2 $H^1$ active contour

Let $M$ denote the set of smooth embedded curves in $\mathbb{R}^3$, which is a differentiable manifold. For $C \in M$, the tangent space of $M$ at $C$ denotes by $T_C M$, which can be seen as the deformation space. Given an energy functional $E(C)$, the process of computation its Gâteaux derivative $\delta E(C, \eta)$ can be expressed as:

$$\delta E(C, \eta) = \lim_{\epsilon \to 0} \frac{E(C + \epsilon \eta) - E(C)}{\epsilon}$$

In order to get the gradient flow of $E(C)$, we must model the deformation space $T_C M$ as an inner product space denoted by $(F, \langle \cdot, \cdot \rangle_F)$. The gradient flow $\nabla_F E(C) \in T_C M$ which is relative to the defined inner product exists when it satisfies the following conditions:

$$\forall \eta \in T_C M, \quad \delta E(C, \eta) = \langle \nabla_F E, \eta \rangle_F \quad \text{and} \quad \nabla_F E \text{ is unique}$$

The inner products for $H^0$ and $H^1$ active contours are defined as following:

$$\langle h, k \rangle_{H^0} := \int_L h(s) \cdot k(s) ds; \quad \langle h, k \rangle_{H^1} := \langle h, k \rangle_{H^0} + \lambda^2 \langle h', k' \rangle_{H^0}$$

where $\lambda \geq 0$ is a weighting coefficient, $L$ is the length of $C$, $h(s)$ and $k(s) \in T_C M$ parameterized by the arclength curve, and the derivatives on $h(s)$ and $k(s)$ are with respect to arclength. At last, the relation between gradient flows for $H^0$ and $H^1$ active contours can be established by solving an ODE derived via relations between $H^0$ and $H^1$ inner products defined above.

$$\nabla_{H^1} E = \nabla_{H^0} E + \sqrt{\lambda} \cdot \frac{s - L/2}{L} \cdot \frac{2L \sqrt{\lambda} \sinh \left( \frac{1}{2 \sqrt{\lambda}} \right)}{2L \sqrt{\lambda} \sinh \left( \frac{1}{2 \sqrt{\lambda}} \right)}$$

## 3 Variance active contour

### 3.1 New inner product for variance active contour

The prior information of active contour with global translation preference can also be interpreted as the velocity field along the evolving contour with equivalency preference (see
Figure 1). When the evolving curve $C$ has a globally pure translation $T(s)=\Delta C(s)$, the variance of velocity field along $C$ is zero. Based on this fact, we define the following inner products to entitle active contour to global translation preference.

$$<h,k>_{H^0} := \langle h, k \rangle_{H^0} + \lambda <h, k - \overline{k}>_{H^0}$$

(5)

$$<h,k>_{H^1} := \overline{h} \cdot \overline{k} + \lambda <h, k - \overline{k}>_{H^1}$$

(6)

$$\overline{h} = \frac{1}{L} \int_0^L h(s) ds; \quad \overline{k} = \frac{1}{L} \int_0^L k(s) ds$$

(7)

It is easy to verify that the above two definition are inner products. We can see that the smaller the norm derived by inner products defined in equations (5) and (6) is. By this feature, it can be induced that variance active contour favors global translation, but not restrict to this motion.

![Figure 1](image)

**Figure 1** Equal velocities along the evolving curve

### 3.2 Gradients of variance active contour

In this subsection, we deduce the relationships between variance and $H^0$ active contours. We can get the following equation via equations (2) and (5).

$$\delta E(C, \eta) = \left< \eta, \nabla_{H^0} E \right>_{H^1} = \left< \eta, \nabla_{H^0} E + \lambda (\nabla_{H^1} E - \nabla_{H^0} E) \right>_{H^0}$$

(8)

If a gradient corresponding to a given inner product exists, it is unique. Thus, we can get the following equation.

$$\nabla_{H^0} E + \lambda (\nabla_{H^1} E - \nabla_{H^0} E) = \nabla_{H^0} E$$

(9)

Using periodic boundary conditions, we can get $\nabla_{H^0} E = \nabla_{H^0} E \ [18]$. Finally the relation between the two gradients can be written as.

$$\nabla_{H^0} E = \frac{\nabla_{H^0} E}{1+\lambda} + \lambda \frac{\nabla_{H^0} E}{1+\lambda}$$

(10)

We can also get the $H^1$ gradient using the similar approach, obtaining that $H^0$ and $H^1$ gradients are the same. As $\lambda \to \infty$, the $H^1$ gradient flow holds globally pure translation equal to the average $H^0$ gradient flow, while it degenerates into the $H^0$ gradient flow as $\lambda \to 0$.

Although the variance and the $H^0$ active contours both favor global translation, the gradient flow of the variance active contour is a simple weighting sum of $H^0$ active contour and corresponding average gradient flow and the gradient flow of $H^1$ is a convolution of $H^0$ active contour and a kernel function.

### 4 Comparison of variance, $H^0$ and $H^1$ active contours

It is easy to see that the norm of the second term relative to inner products for $H^1$ active contour also becomes zero, when the evolving curve only has a globally pure translation. Thus, $H^1$ active contour also prefers global translation. $H^1$ active contour possesses other favorable
properties compared with variance active contour, such as, from coarse to fine motion, much more regularity, but costs are also high, for example, the convolution computation and polygon extraction in the level set framework. By equation (2), we get the following relation.

\[ \langle \tilde{\nabla}_{H^\rho} E, \eta \rangle_{H^\rho} = \langle \tilde{\nabla}_{H^\rho} E, \eta \rangle_{H^\rho} \quad \text{For all} \quad \eta \in T_c M \]  

(11)

Using Parseval's Theorem, we get that.

\[ \tilde{\nabla}_{H^\rho} E(0) = \tilde{\nabla}_{H^\rho} E(0); \quad \tilde{\nabla}_{H^\rho} E(l) = \frac{\tilde{\nabla}_{H^\rho} E(l)}{1 + \lambda^2}, l \in \mathbb{Z} \setminus \{0\} \]  

(12)

where \( \tilde{\cdot} \) represents the Fourier transform. The Fourier series for \( H^1 \) active contour can be written as[18].

\[ \tilde{\nabla}_{H^1} E(0) = \tilde{\nabla}_{H^1} E(0); \quad \tilde{\nabla}_{H^1} E(l) = \frac{\tilde{\nabla}_{H^1} E(l)}{1 + \lambda^2(2\pi l)^2}, l \in \mathbb{Z} \setminus \{0\} \]  

(13)

From equations (12) and (13), we find that, for the zero frequency components of \( H^1 \) and \( H^\rho \) active contours play an more important role than \( H^\rho \) active contour, they have global translation preference. However, except the zero frequency component, the decay coefficients of \( H^1 \) active contour Fourier series are relative to frequency, while the ones of variance active contour are not and only relative to \( \lambda \). This can be interpreted as that we only consider the average perturbation of curves corresponding to the zero frequency. However, the variance active contour possesses better regularity and global property than \( H^\rho \) active contour and has simpler expression than \( H^1 \) active contour inducing fast and easy realization.

5 Numerical experiments

In this section we give some experiments to demonstrate the characteristics of variance active contour. The weighting coefficient \( \lambda \) in equation (5) as well as equation (3) is chosen as 20 in the subsequent numerical experiments, although higher \( \lambda \) entitles them to better global translation preference and produce similar results. The following experiments were realized based on the C-V model [5] which is a region-based model.

Figure 2 shows some snapshots of the evolution of a circular object segmentation in a noisy image, using \( H^1 \), \( H^\rho \) and \( H^\rho \) active contours respectively. The initial contour is also a circle with a translation from the segment object. From Figure 2, we can see that \( H^\rho \) active contour deforms the initial contour arbitrarily to reduce energy giving rise to many unnecessary and unlikely shapes, while \( H^1 \) and \( H^\rho \) active contours with global translation preference, favor preserving the initial shape.

![Figure 2](image) From top to bottom, snapshots of a noisy circular object segmentation using region-based energy by \( H^\rho \) (with a regularization term), \( H^1 \), and \( H^\rho \) active contours.
A plot in Figure 3 shows the roundness of evolving $H^0$, $H^1$, and $H^\lambda$ contours in Figure 2 to measure them how far away from the circular shape. It is easy to see that $H^\lambda$ contour has favorable shape preserving property, but less than $H^1$ contour, while $H^0$ deforms arbitrarily. Firstly the evolving contour is far away from the initial shape, then close to it. However, $H^\lambda$ contour active contour can also obtain the same favorable shape preserving property as $H^1$ active contour by increasing $\lambda$.

![Figure 3](image-url)  
Figure 3  
Roundness of evolving $H^0$, $H^1$, and $H^\lambda$ contours in Figure 2

Table 1 gives the comparisons of Figure 2 among $H^0$, $H^1$ and $H^\lambda$ active contours. From table 1, we can deduce that the iterations of $H^\lambda$ active contour are less than $H^0$ but more than $H^1$ and the consuming time of $H^\lambda$ active contour is the least. This can be interpreted that $H^\lambda$ active contour have less global translation than $H^1$ with the same $\lambda$, while more than $H^0$ active contour and the cost time for each iteration of $H^1$, $H^\lambda$, and $H^0$ active contours is in a decreasing rank.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Iterations</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^0$</td>
<td>77</td>
<td>2.13</td>
</tr>
<tr>
<td>$H^1$</td>
<td>13</td>
<td>0.6</td>
</tr>
<tr>
<td>$H^\lambda$</td>
<td>17</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Table 1 Comparison of Figure 2 among $H^0$, $H^1$, and $H^\lambda$ active contours

In order to show the differences among $H^0$, $H^1$ and $H^\lambda$ active contours, we give another experiment for a circular object detection using region-based energy in a noisy image which contains a small and a big circular objects (see Figure 4). For $H^1$ and $H^\lambda$ active contours with global translation preference, they can leap over many undesired local minima giving expected segmentation results, while $H^0$ active contour was trapped a local minimal, which is unfavorable (see Figure 4). This favorable property is more suitable for object tracking.

![Figure 4](image-url)  
Figure 4  
From left to right, each one respectively represents the initial contour, segmentation results of $H^0$, $H^1$ and $H^\lambda$ active contours

Variance active contour can also be realized exactly using level set method without polygon extraction, which is a complex process and is necessary for $H^1$ active contour. Figure 5 demonstrates the comparisons between region-based $H^0$ and $H^\lambda$ active contours implemented by level set method. It shows that $H^\lambda$ active contour with global translation preference has higher regularity than $H^0$ active contour.
Figure 5: The odd rows represent snapshots of curve evolutions for a noisy square segmentation, driven by region-based $H^u$ active contour with different weighting coefficients of the regularization term ($\mu=0.01 \times 255^2$, $0.1 \times 255^2$, $1.5 \times 255^2$) and $H^u$ active contour without regularization term. The even rows represent the corresponding curves on their level set functions.

Figure 6: Region-based variance active contour without regularization term can cope with topological changes: merging and splitting.

For Variance active contour can be realized directly by level sets, they also can cope with contours topological change easily (see Figure 6). We can find that there is an obviously translation between the first two images in Figure 6, but is not restrict to this translation change.

6 Conclusion

A new inner product was defined based on a prior information that is prior to globally pure translation for curve evolution. Active contour corresponding to the new inner product is named as variance active contour. Then, the gradient flow of variance active contour was deduced. The analysis and the comparisons were given among $H^e$, $H^l$ and variance active contours. By analysis, the prior information for variance and $H^l$ active contours can be considered that the deformation fields along evolving curves are uniform distribution and smoothness respectively. Variance active contour can also be considered as a transition between $H^e$ and $H^l$ active contours, because variance active contour has more favorable features than $H^e$ active contour, but less than $H^l$ active contour. In contrast to $H^l$ active contour, variance active contour can be implemented without polygon extraction using level set method and implemented fast and easily. At last, we simultaneously submit some corresponding supplementary videos to demonstrate and validate variance active contour.
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References


